Notes for ECE 635 (Based on Salehi's Book, Andrews 2005 and my own notes and papers) HTE – 4.10.2011

General

A monochromatic optical wave propagating in free space (vacuum) with constant refractive index spatially and temporally, is given by

$$U\left(\mathbf{R},t\right)=U\left(\mathbf{R}\right)\exp\left(j2\pi ft\right) \qquad \text{(G1)}$$

where $\mathbf{R}=\left(r,\phi,z\right)$ in cylindrical coordinates or $\mathbf{R}=\left(x,y,z\right)$ in Cartesian coordinates. t indicates time dependence. This propagation is governed by the following differential equation

$$\nabla^2 U\left(\mathbf{R},t\right)-c^{-2}\frac{\partial^2 U\left(\mathbf{R},t\right)}{\partial^2 t}=0 \qquad \text{(G2)}$$

where c is the speed of light in free space. Since the time dependence in (G1) is in the form of an exponential function, then (G2) reduces to what is known as Helmholtz equation, expressed as

$$\left(\nabla^2+k^2\right)U\left(\mathbf{R}\right)=0 \qquad \text{(G3)}$$

where $k=2\pi/\lambda$ is the wave number with λ being the wavelength. For the propagation along positive z direction, if we write for the

$$U\left(\mathbf{R}\right)=u\left(\mathbf{r}\right)\exp\left(jkz\right) \qquad \text{(G4)}$$

with $\mathbf{r}=\left(r,\phi\right)$,then (G3) turns into the following paraxial wave equation (PWE), after the omission of the second derivative of $u\left(\mathbf{r}\right)$ with respect to z , i.e., $\frac{\partial^2 u\left(\mathbf{r}\right)}{\partial z^2}\to 0$

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial u\left(\mathbf{r}\right)}{\partial r}\right]+\frac{1}{r^2}\frac{\partial^2 u\left(\mathbf{r}\right)}{\partial^2\phi}+2jk\frac{\partial u\left(\mathbf{r}\right)}{\partial z}=0 \quad , \quad \frac{\partial^2 u\left(\mathbf{r}\right)}{\partial r^2}+\frac{1}{r}\frac{\partial u\left(\mathbf{r}\right)}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u\left(\mathbf{r}\right)}{\partial^2\phi}+2jk\frac{\partial u\left(\mathbf{r}\right)}{\partial z}=0 \qquad \text{(G5)}$$

$$\frac{\partial^2 u(\mathbf{r})}{\partial r_x^2}+\frac{\partial^2 u(\mathbf{r})}{\partial^2 r_y^2}+2jk\frac{\partial u(\mathbf{r})}{\partial z}=0 \qquad \text{PWE in Cartesian coordinates} \qquad \text{(G6)}$$

1. Beam Types

Now we present various beam types on source plane, i.e. $z=0$. For the source coordinate representation, we choose $\mathbf{s}$, so $\mathbf{r}$ is the coordinate for receiver plane, $\mathbf{s}=(s,\phi_s)$ for cylindrical coordinates **,** $\mathbf{s}=(s_x,s_y)$ for Cartesian coordinates . Hence below, $u_s(\ )$ represent the field on the source plane.

1) Fundamental Gaussian Beam

$$u_s(s,\phi_s)=A_c\exp\left(-k\alpha s^2\right) \qquad \text{in cylindrical coordinates} \qquad \text{(1.1a)}$$

$$u_s(s_x,s_y)=A_c\exp\left[-0.5k\left(\alpha_x s_x^2+\alpha_y s_y^2\right)\right] \qquad \text{in Cartesian coordinates} \qquad \text{(1.1b)}$$

where A_c is the amplitude coefficient, $k=2\pi/\lambda$ is the wave number with λ being the wavelength, $\alpha=1/(k\alpha_s^2)+0.5j/F_s$ where α_s and F_s respectively refer to radial Gaussian source size and focusing parameter, $j=\sqrt{-1}$. Similar definitions apply to Cartesian case. Note that in cylindrical coordinates, there is no ϕ_s dependence, thus perfect angular symmetry.

2) Bessel Gaussian Beam

$$u_s(s,\phi_s)=A_c J_n(a_B s)\exp\left[-\left(k\alpha s^2+jn\phi_s\right)\right] \qquad \text{in cylindrical coordinates} \qquad \text{(1.2)}$$

where $J_n(\ )$ is the first kind Bessel function having an order *n*, a_B is the width parameter. The Cartesian coordinate version of Bessel Gaussian beam is rarely used.

3) Modified Bessel Gaussian Beam

$$u_s(s,\phi_s) = A_c I_n(a_B s)\exp\left[-\left(k\alpha s^2 + jn\phi_s\right)\right] \text{ in cylindrical coordinates} \qquad (1.3)$$

where $I_n(\ )$ is the modified Bessel function having an order *n*.

4) Higher Order Sinusoidal Hyperbolic Gaussian Beam

In cylindrical coordinates (without higher order, i.e. Hermite polynomials)

$$u_s(s,\phi_s) = \sum_{\ell=1}^{N} A_\ell \exp\left[-k\alpha_\ell s^2 + \left(\sin\phi_s + \cos\phi_s\right)D_{s\ell}s\right] \qquad (1.4a)$$

In Cartesian coordinates

$$u_s(s_x, s_y) = \sum_{\ell=1}^{N} A_\ell H_{n_\ell}\left(a_{x\ell}s_x + b_{x\ell}\right)\exp\left[-\left(0.5k\alpha_{x\ell}s_x^2 - D_{x\ell}s_x\right)\right]H_{m_\ell}\left(a_{y\ell}s_y + b_{y\ell}\right)\exp\left[-\left(0.5k\alpha_{y\ell}s_y^2 - D_{y\ell}s_y\right)\right] \qquad (1.4b)$$

where D is the displacement parameter, H_n is the Hermite polynomial of order *n*.

5) Laguerre Gaussian beam

In cylindrical coordinates

$$u_s\left(s,\phi_s\right)=A_c\left(\frac{\sqrt{2}s}{\alpha_s}\right)^m\left(-j\right)^m\exp\left(-k\alpha s^2+jm\phi_s\right)L_n^m\left(\frac{2s^2}{\alpha_s^2}\right)\qquad(1.5a)$$

In Cartesian coordinates

$$u_s\left(s_x,s_y\right)=A_c\left[\frac{\left(s_x+js_y\right)}{\alpha_{sc}}\right]^m\left(-j\right)^m\exp\left[-0.5k\left(\alpha_x s_x^2+\alpha_y s_y^2\right)\right]L_n^m\left(\frac{s_x^2}{\alpha_{sx}^2}+\frac{s_y^2}{\alpha_{sy}^2}\right)\qquad(1.5b)$$

where $\alpha_{sc}=\alpha_{sx}=\alpha_{sy}$, $L_n^m(\;)$ is Laguerre polynomial with radial mode number *n* and angular mode number *m*.

6) Higher Order Dark Hollow Beam (incorporating flat topped and annular Gaussian beam)

In cylindrical coordinates (without higher order and rectangular profile)

$$u_s\left(s,\phi_s\right)=\sum_{p=1}^{P}\binom{P}{p}\frac{\left(-1\right)^p}{P}\left[A_1\exp\left(-k\alpha_1 s^2\right)-A_2\exp\left(-k\alpha_2 s^2\right)\right]=\sum_{p=1}^{P}\sum_{i=1}^{2}\binom{P}{p}\frac{\left(-1\right)^p}{P}A_i\exp\left(-k\alpha_i s^2\right)\qquad(1.6a)$$

In Cartesian coordinates

$$u_s\left(s_x, s_y\right) = H_n\left(a_x s_x + b_x\right) H_m\left(a_y s_y + b_y\right) \sum_{p=1}^{P} \sum_{t=1}^{T} \binom{P}{p} \binom{T}{t} \frac{(-1)^{p+t}}{PT} \left\{ A_1 \exp\left[-0.5k\left(p\alpha_{x1} s_x^2 + t_s \alpha_{y1} s_y^2\right)\right] - A_2 \exp\left[-0.5k\left(p\alpha_{x2} s_x^2 + t_s \alpha_{y2} s_y^2\right)\right]\right\} \qquad (1.6b)$$

where a_x and a_y are the width parameters and b_x and b_y are the displacement parameters for the Hermite polynomials H_n and H_m . $\binom{R}{r}$ and $\binom{T}{t}$ are the binomial coefficients associated with the sums taken over *R* and *T* via the indices *r* and *t*. $t_s = t$ when $T > 1$ and $t_s = r$ if $T = 1$. $\alpha_{x1} = 1/\left(k\alpha_{sx1}^2\right) + j/F_{x1}$ with α_{sx1} and F_{x1} are respectively the Gaussian source size and the focusing parameter of the first beam.

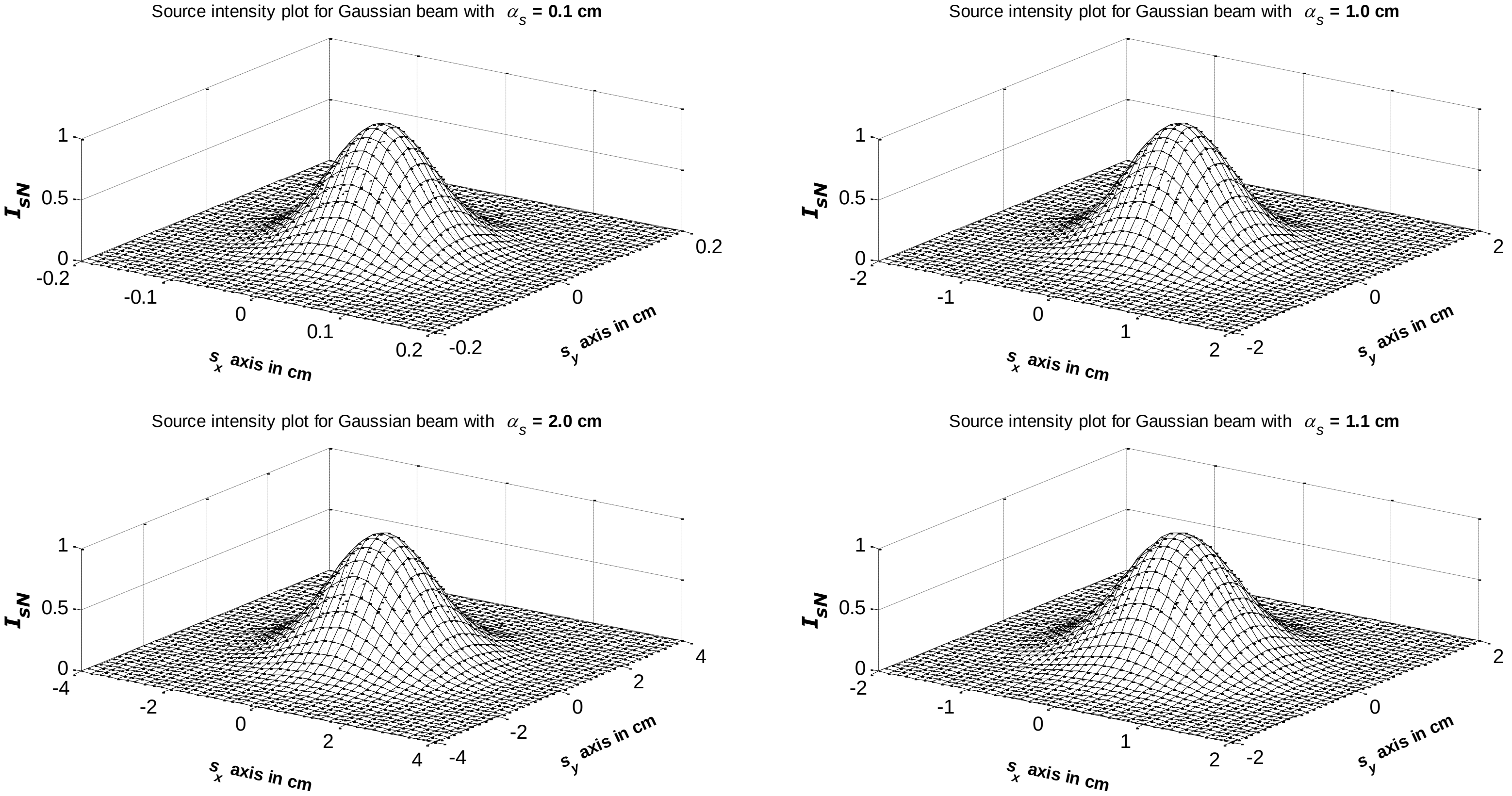
In all cases, intensity will be related to the source field

$$I_s\left(s,\phi_s\right) = u_s\left(s,\phi_s\right) u_s^*\left(s,\phi_s\right) \qquad \text{* indicating conjugate} \qquad (1.7a)$$

$$I_s\left(s_x, s_y\right) = u_s\left(s_x, s_y\right) u_s^*\left(s_x, s_y\right) \qquad (1.7b)$$

$$I_{sN} = I_s(\ )/\max\left[I_s(\ )\right] \qquad (1.7c)$$

Now we illustrate these beams graphically.



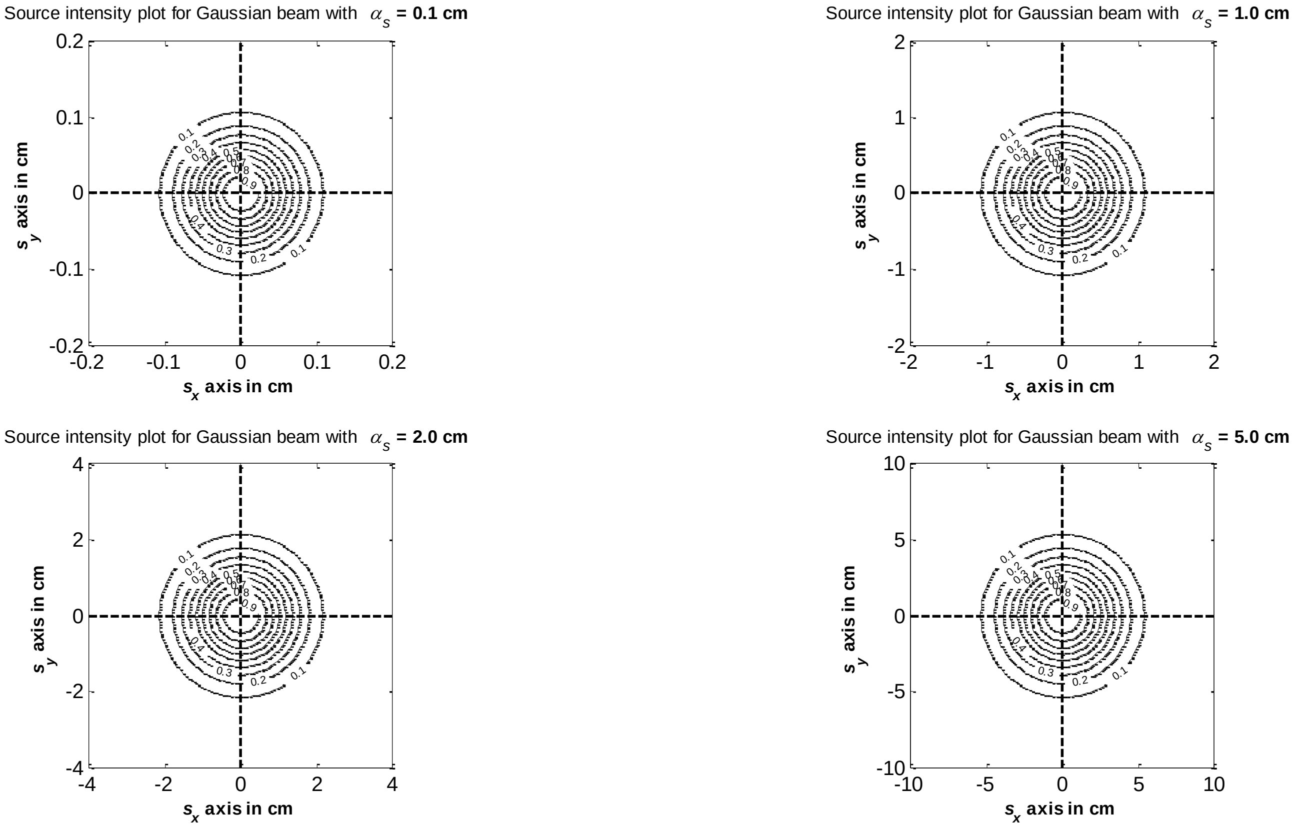
Source intensity plot for Gaussian beam with α_s = 0.1 cm

Source intensity plot for Gaussian beam with α_s = 1.0 cm

Source intensity plot for Gaussian beam with α_s = 2.0 cm

Source intensity plot for Gaussian beam with α_s = 1.1 cm

Fig. 1.1 3D plots of Gaussian beams at different source sizes.



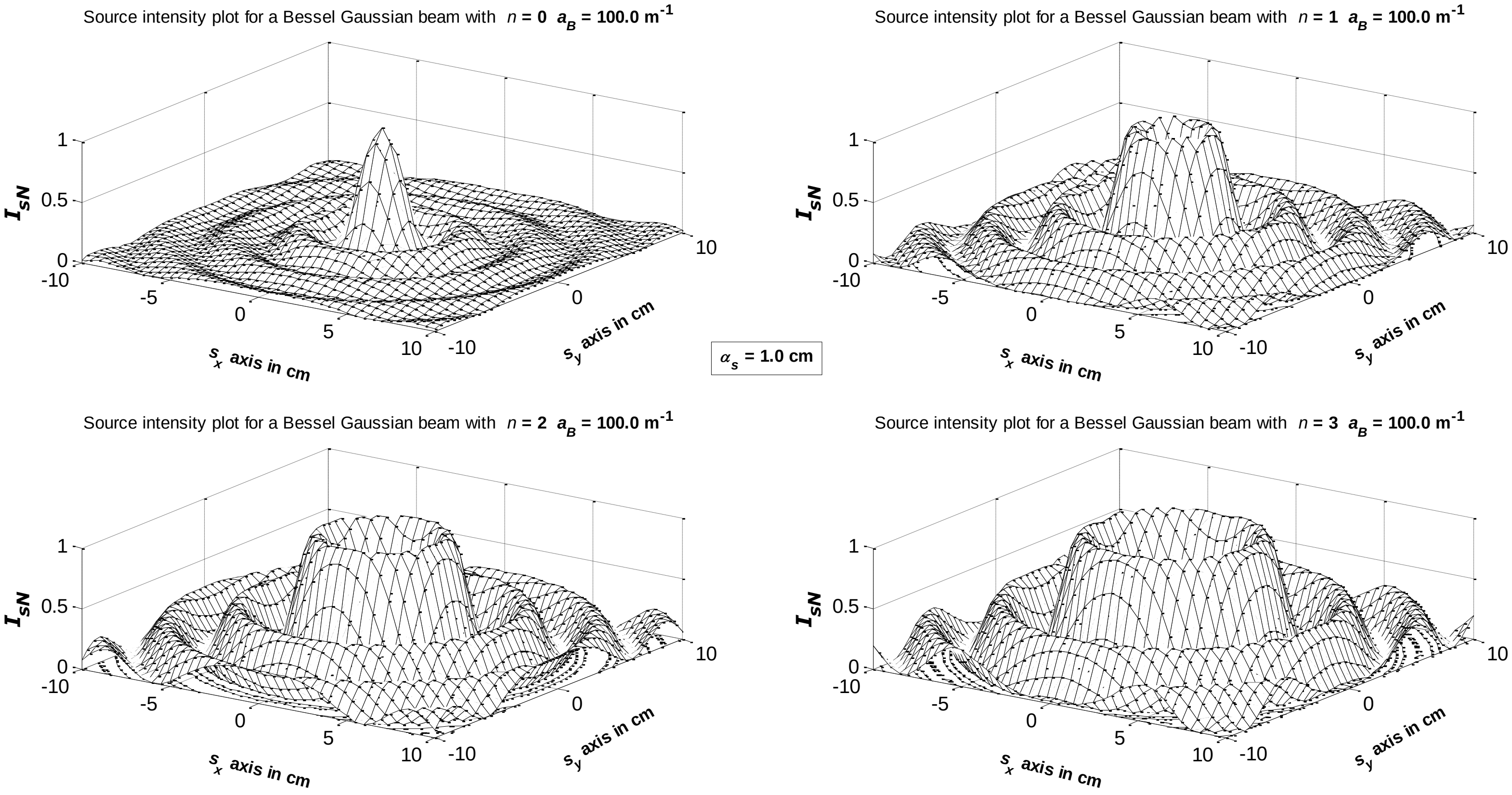
Source intensity plot for Gaussian beam with α_s = **0.1 cm**

Source intensity plot for Gaussian beam with α_s = **1.0 cm**

Source intensity plot for Gaussian beam with α_s = **2.0 cm**

Source intensity plot for Gaussian beam with α_s = **5.0 cm**

Fig. 1.2 Contour plots of Gaussian beams at different source sizes.



Source intensity plot for a Bessel Gaussian beam with n = 0 a_B = 100.0 m^{-1}

Source intensity plot for a Bessel Gaussian beam with n = 1 a_B = 100.0 m^{-1}

Source intensity plot for a Bessel Gaussian beam with n = 2 a_B = 100.0 m^{-1}

Source intensity plot for a Bessel Gaussian beam with n = 3 a_B = 100.0 m^{-1}

Fig. 1.3 3D plots of Bessel Gaussian beams at different beam orders.

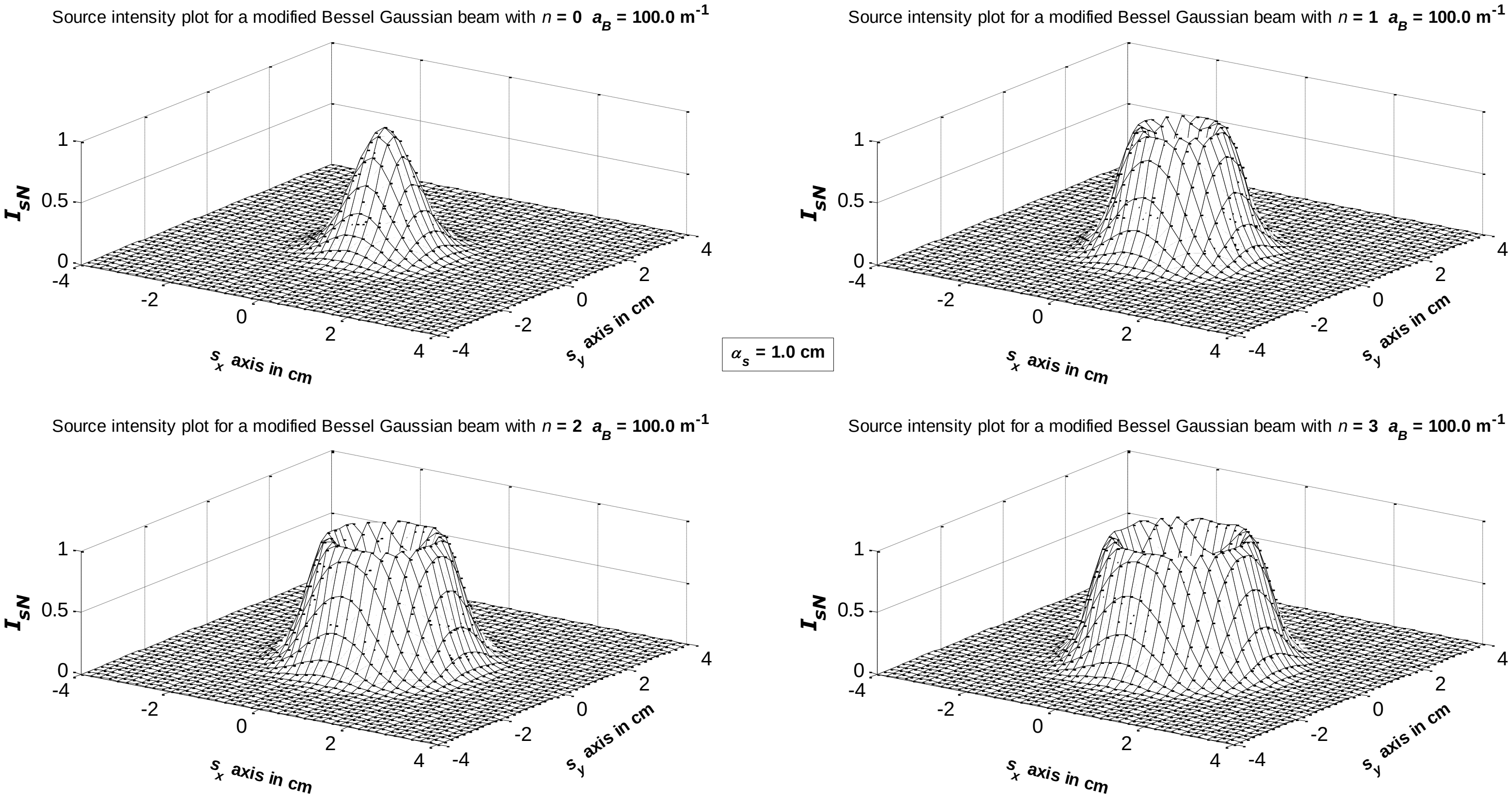


Fig. 1.4 3D plots of modified Bessel Gaussian beams at different beam orders.

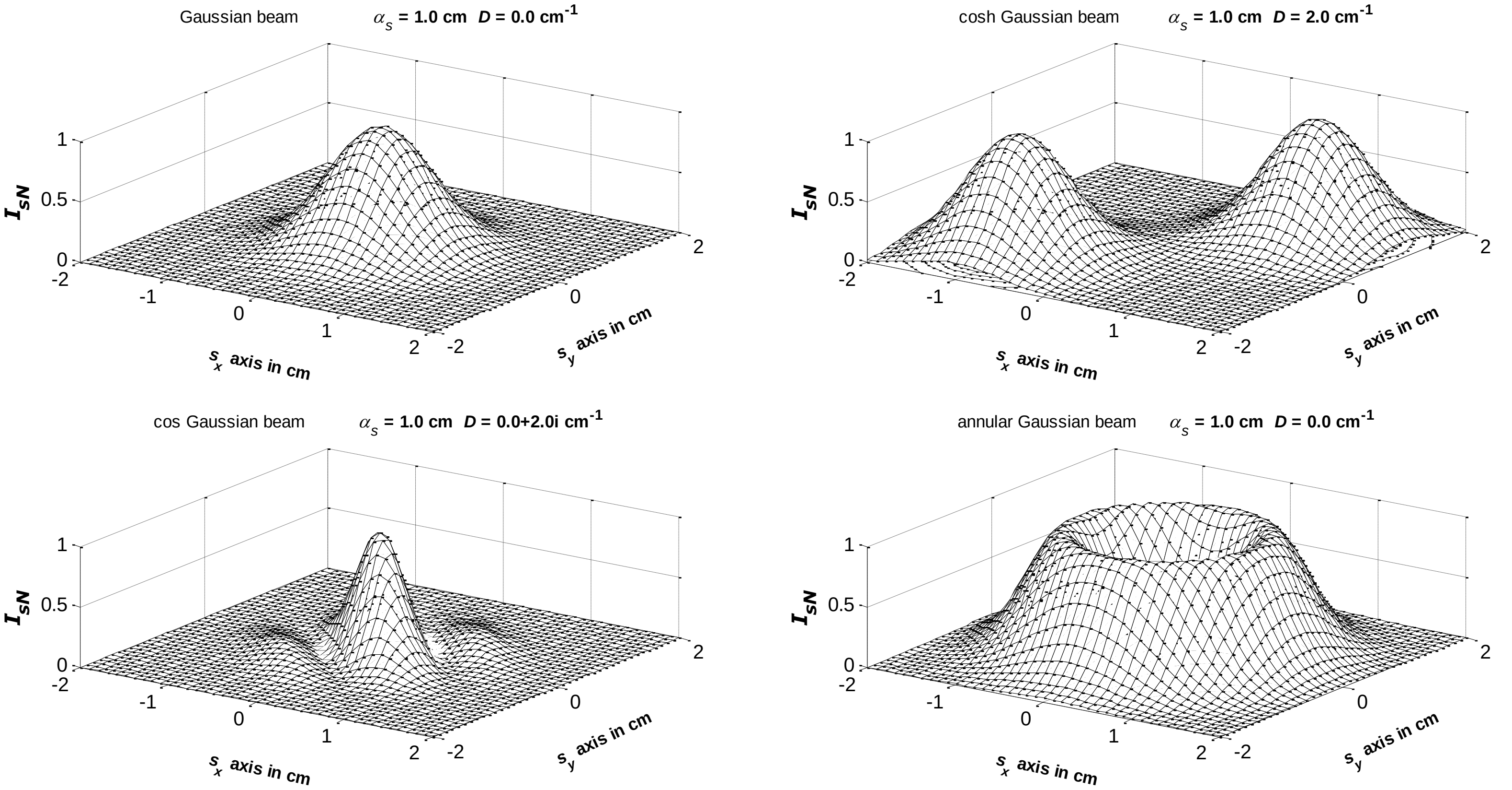


Fig. 1.5 3D plots of sample (lowest order) sinusoidal and hyperbolic Gaussian beams.

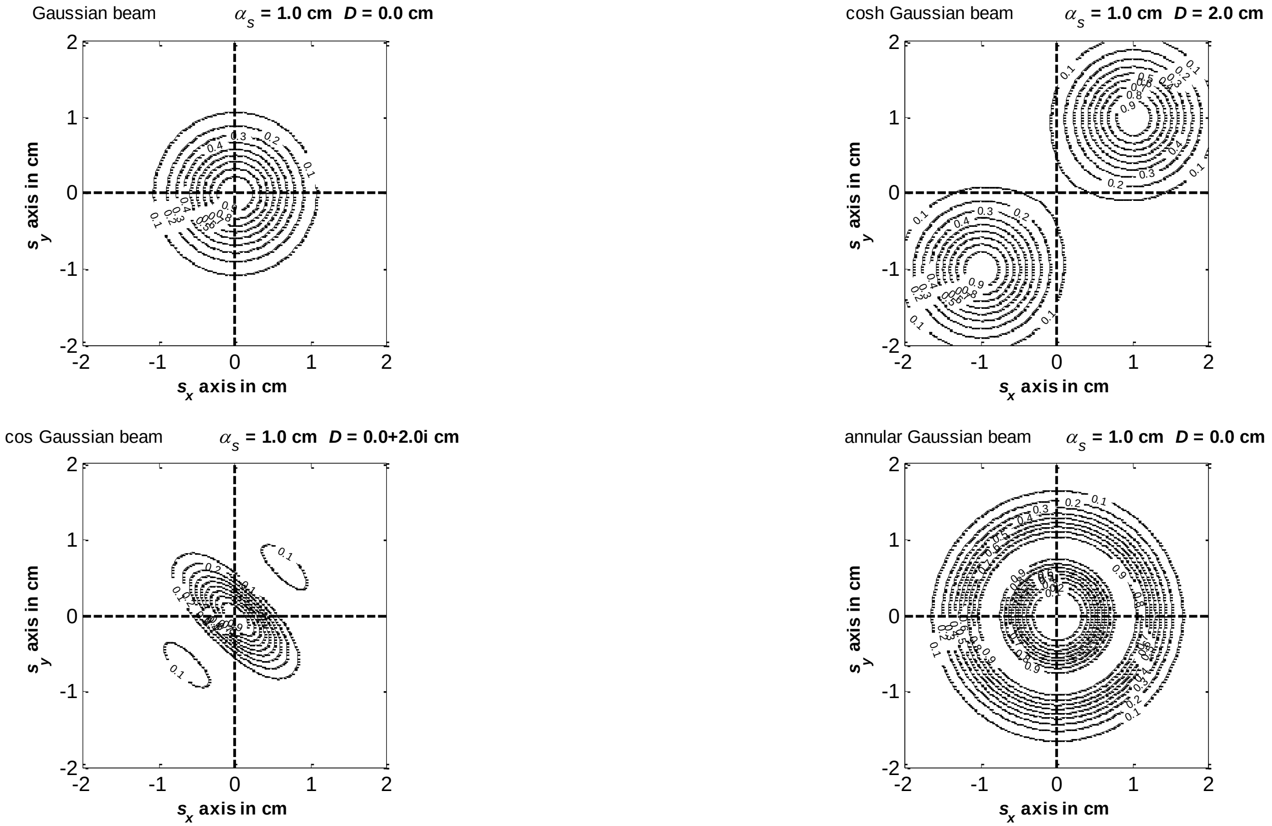


Fig. 1.6 Contour plots of sample (lowest order) hyperbolic sinusoidal Gaussian beams.

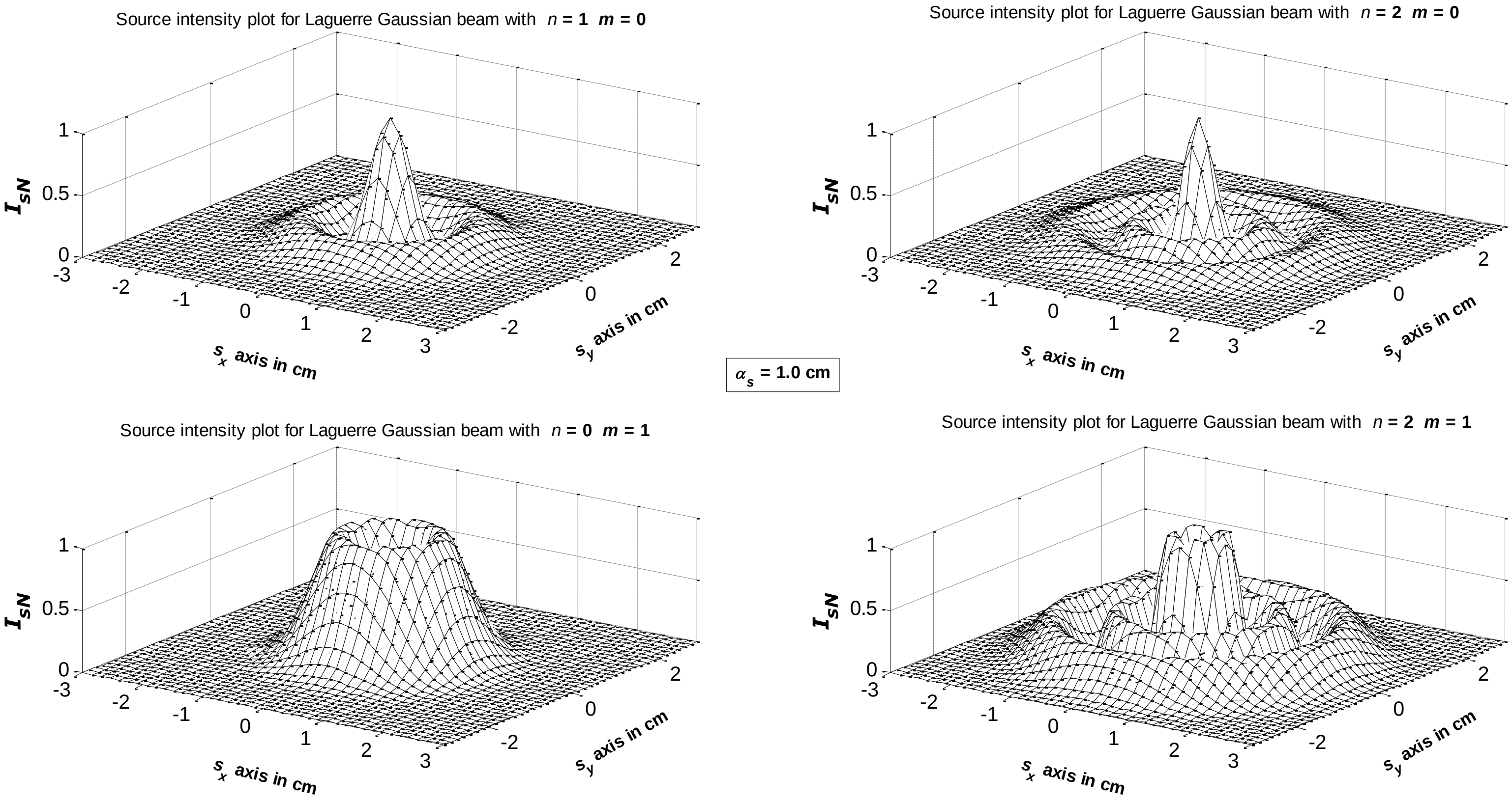


Fig. 1.7 3D plots of sample Laguerre Gaussian beams.

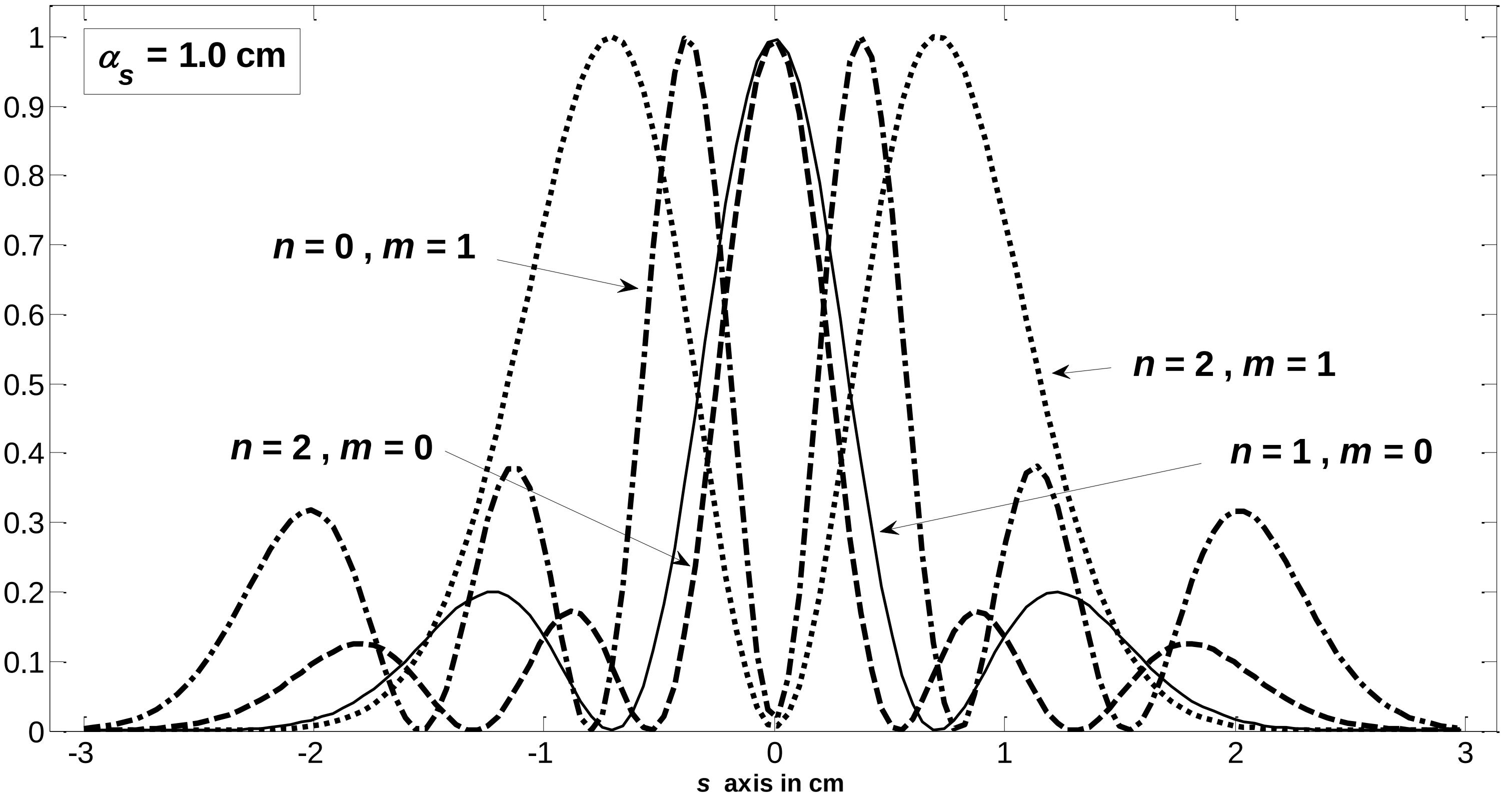


Fig. 1.8 Contour plots of sample Laguerre Gaussian beams.

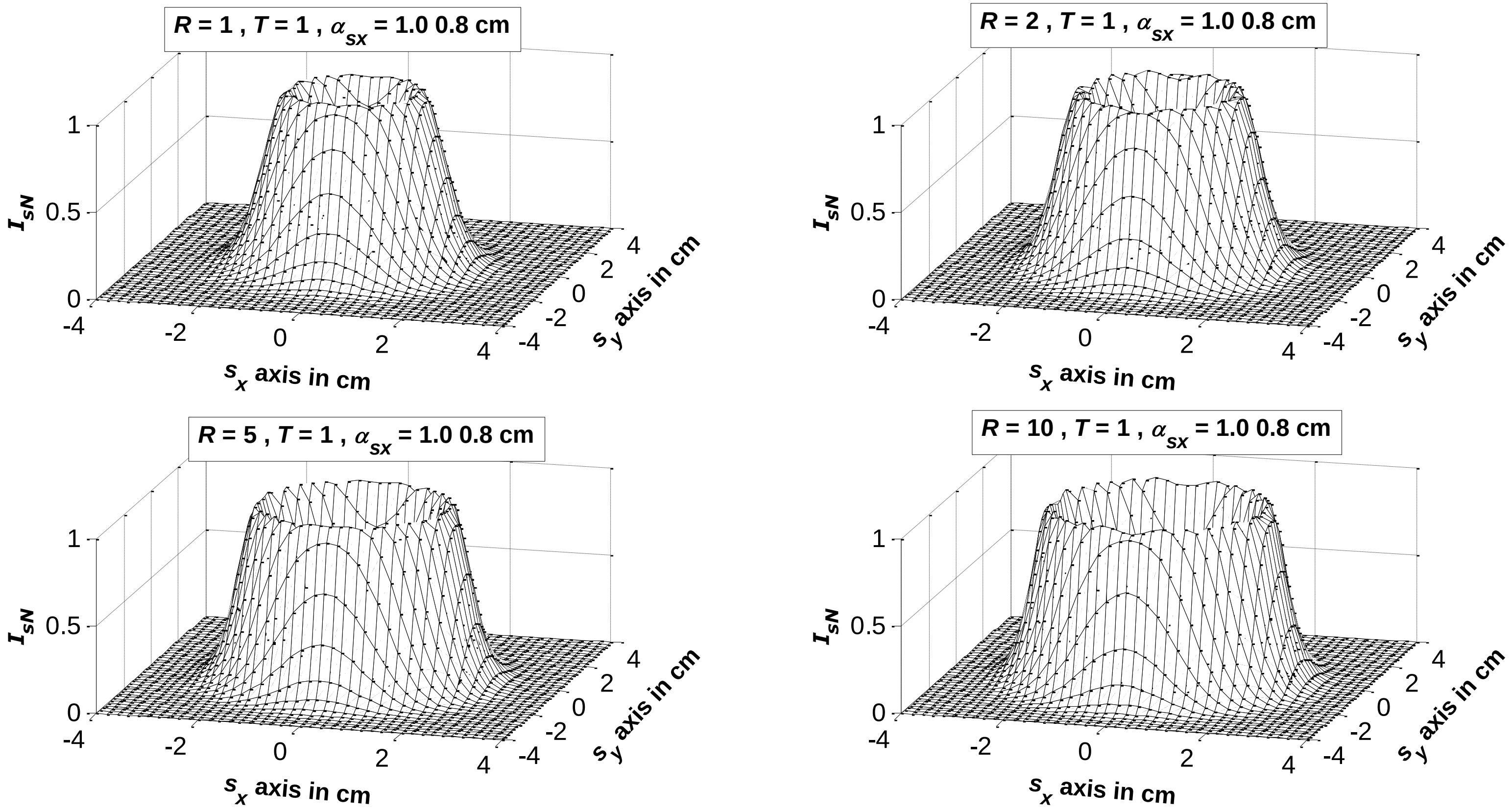


Fig. 1.9 3D plots of sample dark hollow beams (DHBs).

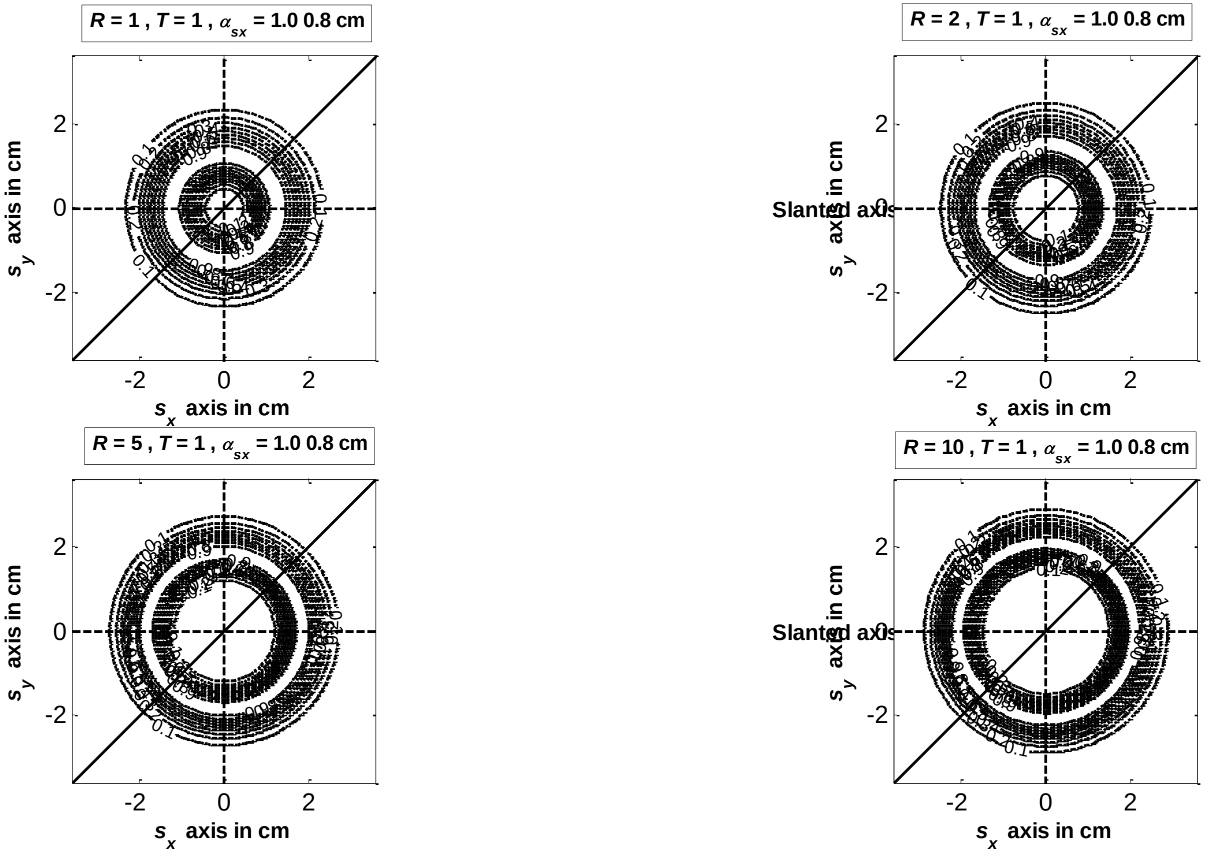


Fig. 1.10 Contour plots of sample dark hollow beams (DHBs).

Exercise 1.1 : The plots in Figs. 1.1 to 1.10 are generated using the Matlab m files, Gaussian_SP.m, Besseljsource.m, Besselisource.m, HypSino_SP.m, Laguerre_SP.m, DHbeam_s.m, available on the course webpage. Using these files identify the corresponding source parameters used in the formulations (1.1) to (1.6) and find how the intensity changes by setting these parameters different from those seen in Figs. 1.1 to 1.10. Record these graphs to print them later as answers to this exercise.

2. Test on PWE

Test on some beam types to see if they satisfy the paraxial wave equation.

A. Fundamental Gaussian Beam : After propagating a distance of z (in free space, i.e. no refractive dependence on spatial and temporal axes), in cylindrical coordinates, the Gaussian beam becomes (excluding $\exp(jkz)$)

$$u_r(\mathbf{r},z) = u_r(r,\phi_r,z) = \frac{A_c}{1+2j\alpha z}\exp\left(-\frac{k\alpha r^2}{1+2j\alpha z}\right) \tag{2.1}$$

To insert into paraxial wave equation, we need to find $\frac{\partial u_r(\mathbf{r},z)}{\partial r}$, $\frac{\partial^2 u_r(\mathbf{r},z)}{\partial r^2}$ and $\frac{\partial u_r(\mathbf{r},z)}{\partial z}$. From (2.1), these are found as

$$\frac{\partial u_r(\mathbf{r},z)}{\partial r} = A_c\frac{-2k\alpha r}{(1+2j\alpha z)^2}\exp\left(-\frac{k\alpha r^2}{1+2j\alpha z}\right) \tag{2.2a}$$

$$\frac{\partial^2 u_r(\mathbf{r},z)}{\partial r^2} = A_c\frac{-2k\alpha}{(1+2j\alpha z)^2}\exp\left(-\frac{k\alpha r^2}{1+2j\alpha z}\right) + A_c\frac{(2k\alpha r)^2}{(1+2j\alpha z)^3}\exp\left(-\frac{k\alpha r^2}{1+2j\alpha z}\right) \tag{2.2b}$$

$$\frac{\partial u_r(\mathbf{r},z)}{\partial z} = A_c\frac{-2j\alpha}{(1+2j\alpha z)^2}\exp\left(-\frac{k\alpha r^2}{1+2j\alpha z}\right) + A_c\frac{2jk\alpha^2 r^2}{(1+2j\alpha z)^3}\exp\left(-\frac{k\alpha r^2}{1+2j\alpha z}\right) \tag{2.2c}$$

Inserting the equations in (2.2) into the paraxial wave equation, i.e.

$$\frac{\partial^2 u_r(\mathbf{r},z)}{\partial r^2} + \frac{1}{r}\frac{\partial u_r(\mathbf{r},z)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u_r(\mathbf{r},z)}{\partial^2\phi} + 2jk\frac{\partial u_r(\mathbf{r},z)}{\partial z} = 0 \tag{2.3}$$

we can easily verify the result.

Example 2.1 : Alternatively, we test PWE in Matlab analytically (symbolically) and numerically as shown below (part of PWETest.m, available on course webpage).

```
clear all;clc ;close all;warning off all
syms a z k r Fir n aB AL Ds as as1 as2
%%%% Field of a Gaussian beam wave after propagating a distance of z, excluding exp(jkz)
Ur = 1/(1 + 2*j*a*z)*exp(-k*a*r^2/(1 + 2*j*a*z));
%%% derivatives of U_r wrt r (first and second orders) and wrt z
Ur1 = diff(Ur,'r');Ur2 = diff(Ur1,'r');Urz = diff(Ur,'z');
Ur1 = diff(Ur,'r');Ur2 = diff(Ur,'r',2);Urz = diff(Ur,'z'); %%% Alternative second derivative
PWE = Ur2 + Ur1/r + 2*j*k*Urz;simplify(PWE)
lamda = 1.55e-6;k = 2*pi/lamda;alfas = 1e-2;a = 1/(k*alfas^2);  %%%% Inserting numeric values
z = 1e3;r = 1e-2;n = 3;Fir = pi/4;aB = 100;%%%% Inserting numeric values
PWE = eval(Ur2 + Ur1/r + 2*j*k*Urz) %%%%% Testing PWE numerically
```

After running the above piece of code, we get PWE = 0.

B. Bessel Gaussian Beam : After propagating a distance of z (in free space, i.e. no refractive dependence on spatial and temporal axes), in cylindrical coordinates, the Bessel Gaussian beam becomes (excluding $\exp(jkz)$ and the amplitude factor, A_c)

$$u_r(\mathbf{r},z) = u_r(r,\phi_r,z) = \frac{-\exp(-jn\phi_r)}{1+2j\alpha z}\exp\left[-\frac{ja_B^2 z + 2\alpha k^2 r^2}{2k(1+2j\alpha z)}\right] J_n\left(\frac{a_B r}{1+2j\alpha z}\right) \qquad (2.4)$$

With the aid of MATLAB code PWETest.m, it is possible to find $\frac{\partial u_r(\mathbf{r},z)}{\partial r}$, $\frac{\partial^2 u_r(\mathbf{r},z)}{\partial r^2}$ and $\frac{\partial u_r(\mathbf{r},z)}{\partial z}$, then by inserting them into PWE again in MATLAB code PWETest.m, we find that PWE is satisfied.

Example 2.2 : Test of PWE for Bessel Gaussian beam of (2.4) is done in the second section of PWETest.m, listed below

```
%% Bessel Gaussian beam
Ur = -exp(-j*n*Fir)/(1 + 2*j*a*z)*exp(-(j*aB^2*z + 2*a*k^2*r^2)/(2*k*(1 + 2*j*a*z)))*besselj(n,aB*r/(1 + 2*j*a*z));
Ur1 = diff(Ur,'r');Ur2 = diff(Urr,'r');UrFi2 = diff(Ur,'Fir',2);Urz = diff(Ur,'z');
```

```
PWE = Ur1 + Ur2/r + UrFi2/r^2 + 2*j*k*Urz;
simplify(PWE)
lamda = 1.55e-6;k = 2*pi/lamda;alfas = 1e-2;a = 1/(k*alfas^2);
z = 1e3;r = 1e-2;n = 3;Fir = pi/4;aB = 100;
PWE = eval(Urrr + Urr/r + UrFi2/r^2 + 2*j*k*Urz)
```

Example 2.3 : By rewriting (G5), with and without $\frac{\partial^2 u\ \mathbf{r}}{\partial^2 z} \to 0$, we get

$$\text{Helmholtz Equation}\ :\ \frac{\partial^2 u(\mathbf{r})}{\partial r^2} + \frac{1}{r}\frac{\partial u(\mathbf{r})}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(\mathbf{r})}{\partial^2 \phi} + \frac{\partial^2 u(\mathbf{r})}{\partial^2 z} + 2jk\frac{\partial u(\mathbf{r})}{\partial z} = 0 \qquad (2.5a)$$

$$\text{Paraxial Wave Equation (PWE)}\ :\ \frac{\partial^2 u(\mathbf{r})}{\partial r^2} + \frac{1}{r}\frac{\partial u(\mathbf{r})}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(\mathbf{r})}{\partial^2 \phi} + 2jk\frac{\partial u(\mathbf{r})}{\partial z} = 0 \qquad (2.5b)$$

As seen in (2.5) and indicated just above (G5), in Helmholtz equation, $\frac{\partial^2 u(\mathbf{r})}{\partial^2 z} \neq 0$, whereas in PWE, $\frac{\partial^2 u(\mathbf{r})}{\partial^2 z} \to 0$, To see the effect of this term, we conduct the following test for Gaussian beam in PWETest.m.

```
Ur = 1/(1 + 2*j*a*z)*exp(-k*a*r^2/(1 + 2*j*a*z));
%%% derivatives of U_r wrt r (first and second orders) and wrt z
Ur1 = diff(Ur,'r');Ur2 = diff(Ur1,'r');Urz = diff(Ur,'z');
Ur1 = diff(Ur,'r');Ur2 = diff(Ur,'r',2);Urz = diff(Ur,'z'); %%% Alternative second derivative
Urz2 = diff(Ur,'z',2); %%% Extra term form Helmholtz Eq
PWE = Ur2 + Ur1/r + 2*j*k*Urz;simplify(PWE)
HE = Ur2 + Ur1/r + Urz2 + 2*j*k*Urz;simplify(HE)
lamda = 1.55e-6;k = 2*pi/lamda;alfas = 1e-2;a = 1/(k*alfas^2);  %%%% Inserting numeric values
z = 1e3;r = 1e-2;n = 3;Fir = pi/4;aB = 100;%%%% Inserting numeric values
PWE = eval(Ur2 + Ur1/r + 2*j*k*Urz) %%%%% Testing PWE numerically
%Ur2s = num2str(eval(Ur2));
['Ur2 = ' num2str(eval(Ur2))]
['Ur1/r = ' num2str(eval(Ur1/r))]
```

```
['Urz2 = ' num2str(eval(Urz2))]
['2jkUrz = ' num2str(eval(2*j*k*Urz))]
```

After running the above piece of code, we get

```
Ur2 = 413.3556+636.1574i

Ur1/r = 628.4353+425.0037i

Urz2 = 3.3446e-007-1.3213e-007i

2jkUrz = -1041.7909-1061.1611i
```

where, Ur2 $\rightarrow \dfrac{\partial^2 u(\mathbf{r})}{\partial r^2}$, Ur1/r $\rightarrow \dfrac{1}{r}\dfrac{\partial u(\mathbf{r})}{\partial r}$, Urz2 $\rightarrow \dfrac{\partial^2 u(\mathbf{r})}{\partial^2 z}$, 2jkUrz $\rightarrow 2jk\dfrac{\partial u(\mathbf{r})}{\partial z}$ (2.6)

As seen, the numeric value of $\dfrac{\partial^2 u(\mathbf{r})}{\partial^2 z}$ is too small compared with the others, hence it is disregarded in PWE. Note that in the above case, Gaussian beam is used, thus there is no variation with respect to ϕ, this way, the term $\dfrac{\partial^2 u(\mathbf{r})}{\partial^2 \phi}$ is zero.

Exercise 2.1 : Prove that the Cartesian coordinate equivalence of the Gaussian beam given in (2.7) satisfies the PWE,either by modifying PWETest.m or by writing your own code.

$$u_r(r_x, r_y, z) = \frac{1}{(1+j\alpha_x z)^{0.5}(1+j\alpha_y z)^{0.5}} \exp\left[-\frac{k\alpha_x r_x^2}{2(1+j\alpha_x z)} - \frac{k\alpha_y r_y^2}{2(1+j\alpha_y z)}\right] \qquad (2.7)$$

Hint : Use (G6) given above, which is $\dfrac{\partial^2 u(\mathbf{r})}{\partial r_x^2} + \dfrac{\partial^2 u(\mathbf{r})}{\partial^2 r_y^2} + 2jk\dfrac{\partial u(\mathbf{r})}{\partial z} = 0$

Exercise 2.2 : The final part of m file, PWETest.m is devoted to testing (lowest order) sinusoidal and hyperbolic Gaussian beam of (1.4a). Run PWETest.m to see that the propagating versions of various beams contained in (1.4a) satisfy PWE.

3. Huygens Fresnel (HF) Integral

This integral is used to find the received field from a given source field. Below we give the expression for cylindrical as well as Cartesian coordinates

$$u_r\left(r,\phi_r,z\right)=\frac{-jk\exp\left(jkz\right)}{2\pi z}\int\limits_0^{\infty}\int\limits_0^{2\pi} ds d\phi_s s u_s\left(s,\phi_s\right)\exp\left\{\frac{jk}{2z}\left[-2rs\cos\left(\phi_r-\phi_s\right)+s^2+r^2\right]\right\} \quad \text{in cylindrical coordinates} \qquad (3.1a)$$

$$u_r\left(r_x,r_y,z\right)=\frac{-jk\exp\left(jkz\right)}{2\pi z}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} ds_x ds_y u_s\left(s_x,s_y\right)\exp\left\{\frac{jk}{2z}\left[-2s_xr_x-2s_yr_y+s_x^2+s_y^2+r_x^2+r_y^2\right]\right\} \quad \text{in Cartesian coordinates} \qquad (3.1b)$$

where the exp term in the integrand is the paraxially approximated Green’s function for spherical source. Note that HF integral in Cartesian coordinates is uncoupled, that is *x* and *y* are separable, whereas in cylindrical coordinates, coupling exists between s and ϕ_s . This way HF integral of cylindrical coordinates is a double integral, but if the source field expression is also separable in s_x and s_y, that is $u_s\left(s_x,s_y\right)=u_s\left(s_x\right)u_s\left(s_y\right)$ then HF integral in Cartesian coordinates actually into the multiplication of two individual one fold integrals as shown below

$$u_r\left(r_x,r_y,z\right)=\frac{-jk\exp\left(jkz\right)}{2\pi z}\exp\left[\frac{jk}{2z}\left(r_x^2+r_y^2\right)\right]\int\limits_{-\infty}^{\infty} ds_y u_s\left(s_y\right)\exp\left[\frac{jk}{2z}\left(-2s_yr_y+s_y^2\right)\right]\int\limits_{-\infty}^{\infty} ds_x u_s\left(s_x\right)\exp\left[\frac{jk}{2z}\left(-2s_xr_x+s_x^2\right)\right] \qquad (3.2)$$

Note that HF integral corresponds to convolution integral, in this sense it transforms beam on the source plane onto receiver plane.

Sample derivation for Gaussian beam

Gaussian beam expressions on source plane in both coordinates are

$$u_s\left(s,\phi_s\right)=A_c\exp\left(-k\alpha s^2\right) \quad \text{in cylindrical coordinates} \qquad (3.3a)$$

$$u_s\left(s_x,s_y\right)=A_c\exp\left[-0.5k\left(\alpha_x s_x^2+\alpha_y s_y^2\right)\right] \quad \text{in Cartesian coordinates} \qquad (3.3b)$$

By taking the cylindrical coordinates expression and inserting it into (3.1a), first we solve the integration over ϕ_s using

$$\int_0^{2\pi} d\phi_s \exp\left[-\frac{jk}{z} rs\cos\left(\phi_r - \phi_s\right)\right] = 2\pi J_0\left(\frac{krs}{z}\right) \quad (3.4)$$

This is a reduced version of 3.937.2 of Gradshteyn and Ryzhik 2007, pp. 496

Then the remaining part of the integral over s is in the following form and can be solved via

$$\int_0^{\infty} sdsJ_0\left(\frac{krs}{z}\right)\exp\left[\left(-k\alpha + \frac{jk}{2z}\right)s^2\right] = \frac{z}{2k\alpha z - jk}\exp\left[-\frac{kr^2}{z\left(4\alpha z - 2j\right)}\right] \quad (3.5)$$

This is a reduced version of 6.631.4 of Gradshteyn and Ryzhik 2007, pp. 706

Collecting all terms and simplifying, the received field of Gaussian beam will be

$$u_r\left(r,\phi_r,z\right) = A_c \frac{\exp\left(jkz\right)}{1+2j\alpha z}\exp\left(-\frac{k\alpha r^2}{1+2j\alpha z}\right) \quad (3.6)$$

Note $\exp\left(jkz\right)$ of Green's function is included

Now we work in Cartesian coordinates and apply HF integral (3.1b) to (3.3b). Take integration over s_x which can be solved as follows

$$\int_{-\infty}^{\infty} ds_x \exp\left[\left(-0.5k\alpha_x + \frac{jk}{2z}\right)s_x^2 - \frac{jk}{z}r_x s_x\right] = \left(\frac{2z\pi}{k\alpha_x z - jk}\right)^{0.5}\exp\left[-\frac{kr_x^2}{2z\left(\alpha_x z - j\right)}\right] \quad (3.7)$$

This is 3.323.2 of Gradshteyn and Ryzhik 2007, pp. 337

Doing the same for s_y, collecting other terms we get

$$u_r\left(r_x,r_y,z\right) = A_c \frac{\exp\left(jkz\right)}{\left(1+j\alpha_x z\right)^{0.5}\left(1+j\alpha_y z\right)^{0.5}}\exp\left[-\frac{k\alpha_x r_x^2}{2\left(1+j\alpha_x z\right)} - \frac{k\alpha_y r_y^2}{2\left(1+j\alpha_y z\right)}\right] \quad (3.8)$$

(3.8) will become the same as (3.6) at $\alpha_x = \alpha_y = 2\alpha,\ r_x^2 + r_y^2 = r^2$.

Exercise 3.1 : Verify (3.6) and (3.8) by showing the intermediate steps. Also verify that $\alpha_x = \alpha_y = 2\alpha,\ r_x^2 + r_y^2 = r^2$, (3.6) and (3.8) are identical.

Exercise 3.2 : From (3.6) and (3.8), find $I_r\left(r,\phi_r,z\right) = u_r\left(r,\phi_r,z\right)u_r^*\left(r,\phi_r,z\right)$ and $I_r\left(r_x,r_y,z\right) = u_r\left(r_x,r_y,z\right)u_r^*\left(r_x,r_y,z\right)$.

Exercise 3.3 : Plot the 3D graphs of $I_r\left(r,\phi_r,z\right)=u_r\left(r,\phi_r,z\right)u_r^*\left(r,\phi_r,z\right)$ and $I_r\left(r_x,r_y,z\right)=u_r\left(r_x,r_y,z\right)u_r^*\left(r_x,r_y,z\right)$, benefitting from Gaussian_SP.m.

4. ABCD Formalism

ABCD law defines in a matrix fashion, the combined transfer function of a propagating medium including optical elements on the way. For instance, for the setup given in Fig. 4.1, the propagation path contains no optical elements

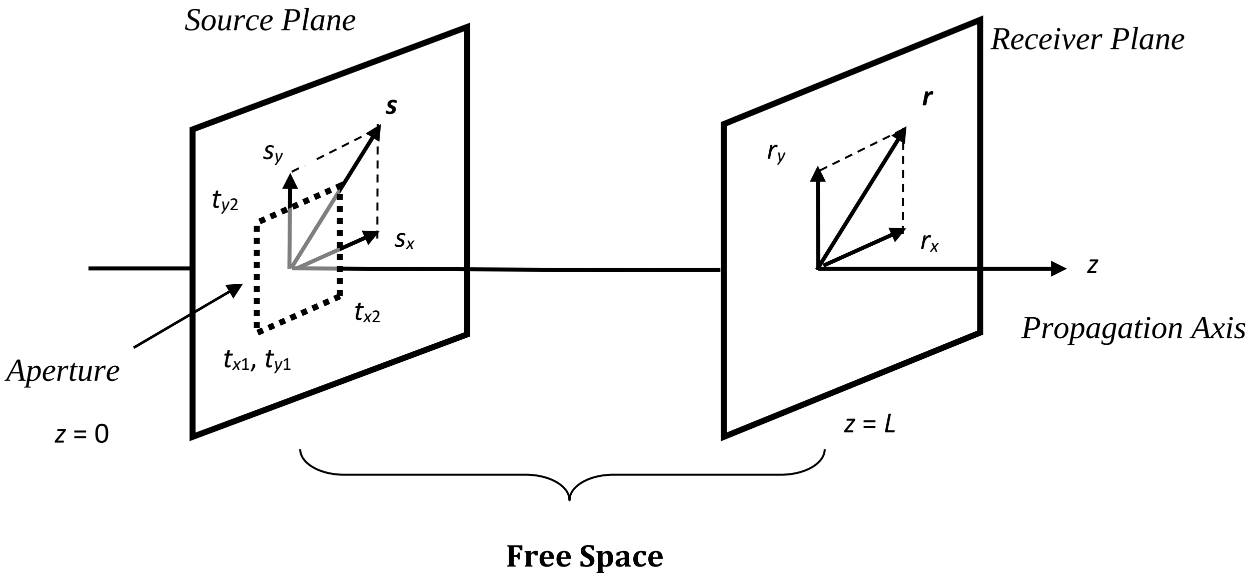


Fig. 4.1 Illustration of free space propagation without an optical element on the path.

Hence

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L \\ 0 & 1 \end{pmatrix} \tag{4.1}$$

For the general form of ABCD matrix, Huygens Fresnel (also known as Collins integral) integral will be in Cartesian coordinates, with a rectangular aperture placed on the source plane with the dimensions shown in Fig. 4.1.

$$u_r\left(\mathbf{r}, z=L\right) = \frac{-jk\exp\left(jkz\right)}{2\pi B}\int_{t_{y1}}^{t_{y2}}\int_{t_{x1}}^{t_{x2}} \boldsymbol{d^2 s} u_s\left(\mathbf{s}\right)\exp\left\{\frac{jk}{2B}\left[A\left(s_x^2+s_y^2\right)-2\left(s_x r_x + s_y r_y\right)+D\left(r_x^2+r_y^2\right)\right]\right\} \tag{4.2}$$

This way *B* represents *L*, while *C* is not used. It is easy to see that at the limit of $A=1,\ B=L,\ D=1,\ t_{x1},\ t_{y1}\rightarrow-\infty,\ t_{x2},\ t_{y2}\rightarrow\infty$, (4.2) will become identical to (3.1b).

The case of having an optical element on the way is illustrated below

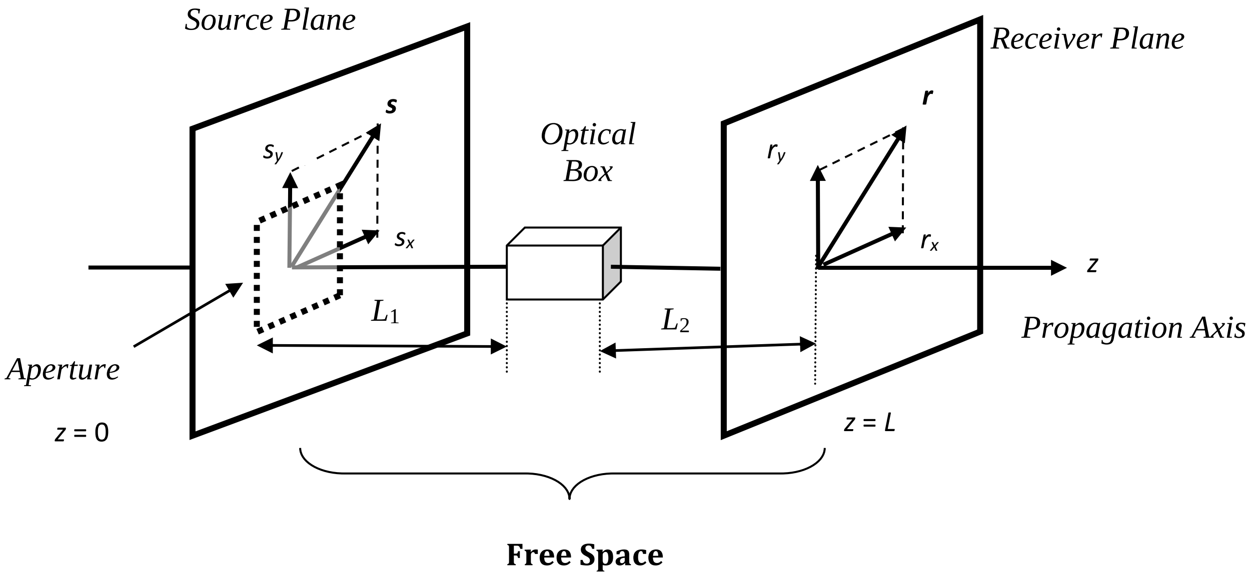


Fig. 4.2 Illustration of free space propagation with an optical element (optical box) on the path.

Assuming the optical box is a thin (which means the lens occupies zero length along propagation axis) lens and denoting the transmission apertures and the focal length of this lens as α_a and F_a such that $\alpha_G = 1/(k\alpha_a^2) + j/F_a$, we get the following ABCD matrix for this lens

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ j\alpha_G & 1 \end{pmatrix} \qquad (4.3)$$

Including the sections L_1 and L_2, the whole ABCD matrix from $z = 0$ upto $z = L$ is obtained as

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 1 & L_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ j\alpha_G & 1 \end{pmatrix} \begin{pmatrix} 1 & L_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 + j\alpha_{lx} L_2 & L_1 + L_2\left(1 + j\alpha_{lx} L_1\right) \\ j\alpha_{lx} & 1 + j\alpha_{lx} L_1 \end{pmatrix} \qquad (4.4)$$

As seen from (4.4), the multiplication is carried out in the reverse order.

Upon selecting the source beam to be the higher order sinusoidal hyperbolic beam, $u_r\left(\mathbf{r}, z = L\right)$ turns into (6) of the following reference

H. T. Eyyuboğlu, "Hermite hyperbolic / sinusoidal Gaussian beams in *ABCD* systems", *Optik* **118**(6), 289-295 (2007).

5. Partially Coherent Sources

If a source has phase discontinuities either against time (temporal) or spatial coordinate axis, the it is said to be partially coherent.

Expressing these phase discontinuities in spatial domain, we write

$$\bar{u}_s\left(\mathbf{s}\right) = u_s\left(\mathbf{s}\right)\exp\left[j\vartheta\left(\mathbf{s}\right)\right] \qquad (5.1)$$

This way $u_s\left(\mathbf{s}\right)$ corresponds to a fully coherent source, while $\bar{u}_s\left(\mathbf{s}\right)$ to partially coherent source. Considering that phase is meaningful, when measured over two points, we define mutual coherence function Γ and take the average of spatial variations of phase into an exponential function as follows

$$\begin{aligned}\Gamma_s(\mathbf{s}_1,\mathbf{s}_2) &= u_s(\mathbf{s}_1)u_s^*(\mathbf{s}_2)\left\langle \exp\left\{\left[j\vartheta(\mathbf{s}_1)-j\vartheta(\mathbf{s}_2)\right]\right\}\right\rangle \\ &= u_s(\mathbf{s}_1)u_s^*(\mathbf{s}_2)\exp\left(-\frac{|\mathbf{s}_1-\mathbf{s}_2|^2}{2\sigma_s^2}\right) \qquad (5.2)\end{aligned}$$

where σ_s is the measure of spatial coherence of the source such that as $\sigma_s \to \infty$ the source becomes fully coherent and when $\sigma_s \to 0$, the source will become fully incoherent. But for σ_s to be effective, it has to be comparable to source size of the beam. Defining partial coherence in the manner shown in (5.2) is known as Gaussian Schell Model (GSM).

It is possible to find the mutual coherence function on the receiver plane by using Huygens Fresnel integral in the following manner

$$\Gamma_r(\mathbf{r}_1,\mathbf{r}_2,L) = k^2/(2\pi L)^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d^2\mathbf{s}_1 d^2\mathbf{s}_2 \Gamma_s(\mathbf{s}_1,\mathbf{s}_2)\exp\left\{jk\left[|\mathbf{r}_1-\mathbf{s}_1|^2-|\mathbf{r}_2-\mathbf{s}_2|^2\right]/(2L)\right\} \qquad (5.3)$$

(5.3) will become a quadruple integral if *x* and *y* terms are written in full

$$\begin{aligned}\Gamma_r(r_{1x},r_{1y},r_{2x},r_{2y},L) &= \left(\frac{k}{2\pi L}\right)^2 \exp\left[\frac{jk}{2L}\left(r_{1x}^2+r_{1y}^2-r_{2x}^2-r_{2y}^2\right)\right]\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ds_{1x}ds_{1y}ds_{2x}ds_{2y}\Gamma_s(s_{1x},s_{1y},s_{2x},s_{2y}) \\ &\times\exp\left[\frac{jk}{2L}\left(s_{1x}^2-2r_{1x}s_{1x}+s_{1y}^2-2r_{1y}s_{1y}-s_{2x}^2+2r_{2x}s_{2x}-s_{2y}^2+2r_{2y}s_{2y}\right)\right] \qquad (5.4)\end{aligned}$$

Note that $\Gamma_r \;\; r_{1x}, r_{1y}, r_{2x}, r_{2y}, L$ is complex as seen from (5.4), but at the setting of $\mathbf{r}_1 = \mathbf{r}_2$, mutual coherence function becomes real and equal to intensity, thus

$$I_r(\mathbf{r},L) = \Gamma_r(\mathbf{r},\mathbf{r},L) = \Gamma_r(r_x,r_y,r_x,r_y,L) \qquad (5.5)$$

As lab experiment we take m file called Transmittance635.m and find the intensity profiles of partially coherent Gaussian, sinusoidal hyperbolic Gaussian and annular Gaussian beams at different settings of partial coherence parameter, propagation lengths.

Example 5.1 : In Figs. 5.1 and 5.2, we provide 3D and contours plots of such beams at selected source and propagation parameters.

Exercise 5.1 : Using Transmittance635.m set the source and propagation parameters different from those listed in Figs. 5.1 and 5.2. Obtain plots of these beams and comment on how the beam profile changes with the variations in these parameters.

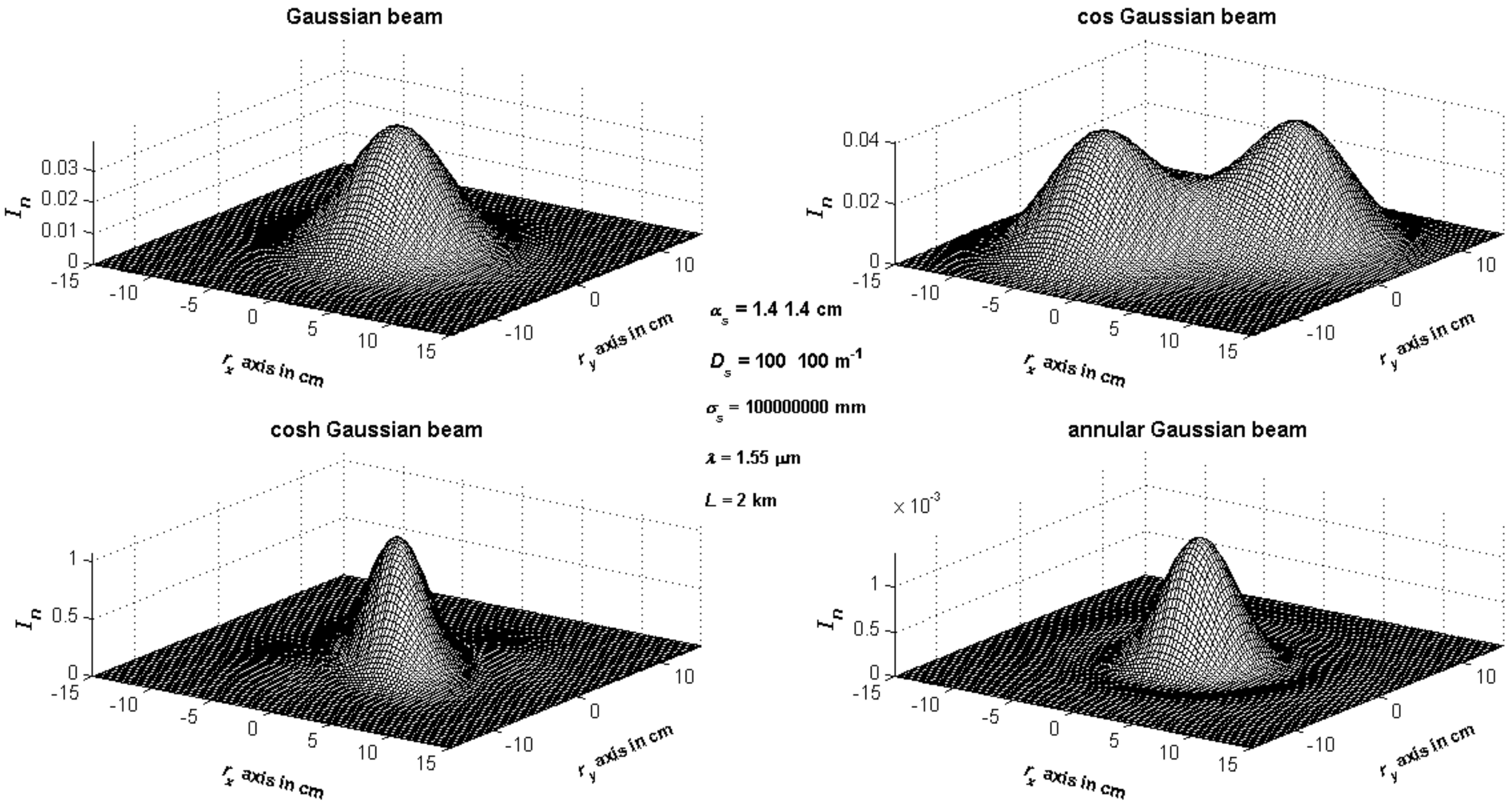


Fig. 5.1 3D plots of sample partially coherent Gaussian, sinusoidal hyperbolic Gaussian and annular Gaussian beams.

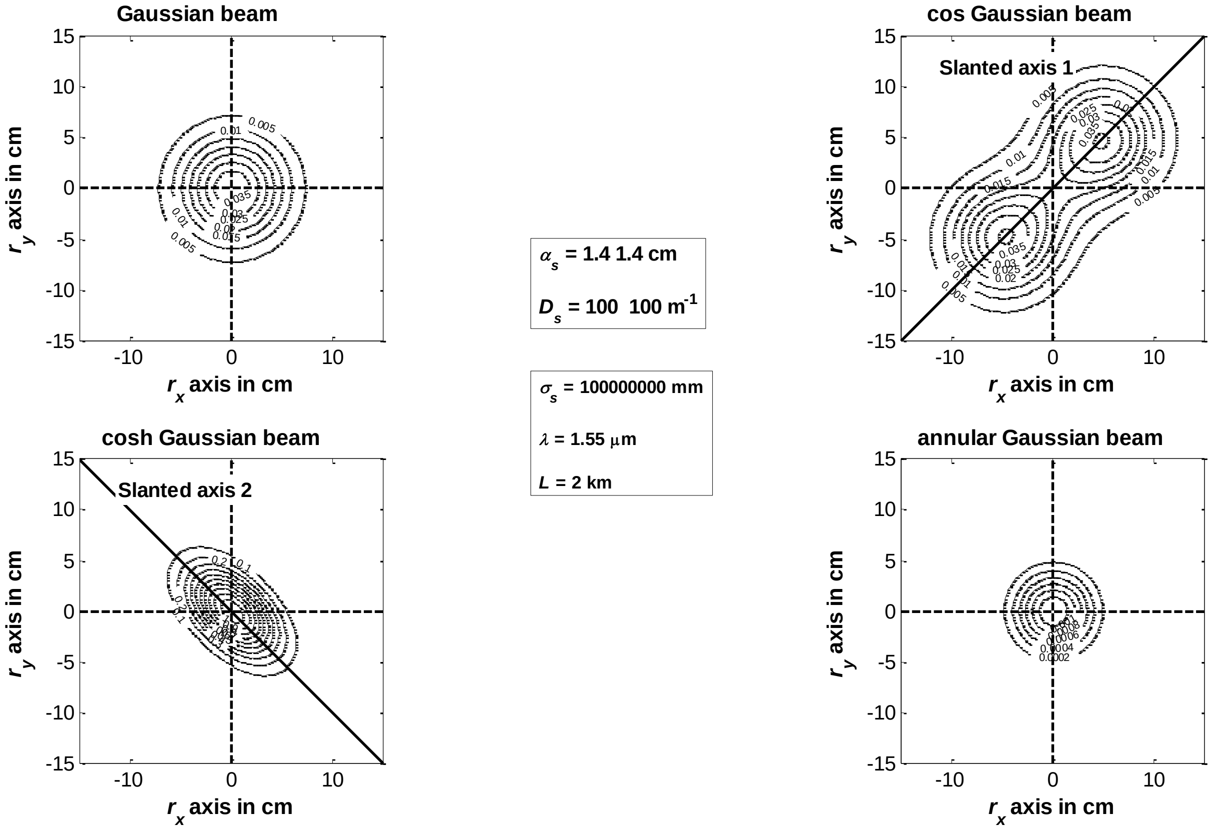


Fig. 5.2 Contour plots of sample partially coherent Gaussian, sinusoidal hyperbolic Gaussian and annular Gaussian beams.

Sample derivation for Sinusoidal Hyperbolic Gaussian beam

By setting the source beam in cylindrical and Cartesian coordinates as follows (excluding the higher order property)

$$u_s\left(s,\phi_s\right) = \sum_{\ell=1}^{N} A_\ell \exp\left[-k\alpha_\ell s^2 + \left(\sin\phi_s + \cos\phi_s\right) D_{s\ell} s\right] \qquad \text{in cylindrical coordinates} \qquad (5.6)$$

$$u_s\left(s_x,s_y\right)=\sum_{\ell=1}^{N}A_\ell\exp\left[-\left(0.5k\alpha_{x\ell}s_x^2-D_{x\ell}s_x\right)\right]\exp\left[-\left(0.5k\alpha_{y\ell}s_y^2-D_{y\ell}s_y\right)\right]\qquad\text{in Cartesian coordinates}\qquad(5.7)$$

Then adding partial coherence as described above, the mutual coherence functions will become

$$\Gamma_s\left(s_1,\phi_{s_1},s_2,\phi_{s_2}\right)=\exp\left[-\frac{s_1^2+s_2^2-2s_1s_2\cos\left(\phi_{s_1}-\phi_{s_2}\right)}{2\sigma_s^2}\right]\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}A_{\ell_1}A_{\ell_2}\exp\left[-k\alpha_{\ell_1}s_1^2-k\alpha_{\ell_2}^{*}s_2^2+\left(\sin\phi_{s_1}+\cos\phi_{s_1}\right)D_{s\ell_1}s_1+\left(\sin\phi_{s_2}+\cos\phi_{s_2}\right)D_{s\ell_2}^{*}s_2\right]$$

$$\text{in cylindrical coordinates}\qquad(5.8)$$

$$\Gamma_s\left(s_{1x},s_{1y},s_{2x},s_{2y}\right)=\exp\left(-\frac{s_{1x}^2+s_{2x}^2+s_{1y}^2+s_{2y}^2-2s_{1x}s_{2x}-2s_{1y}s_{2y}}{2\sigma_s^2}\right)\sum_{\ell_1=1}^{N}A_{\ell_1}\exp\left[-\left(0.5k\alpha_{x\ell_1}s_{1x}^2-D_{x\ell_1}s_{1x}\right)\right]\exp\left[-\left(0.5k\alpha_{y\ell_1}s_{1y}^2-D_{y\ell_1}s_{1y}\right)\right]$$
$$\times\sum_{\ell_2=1}^{N}A_{\ell_2}^{*}\exp\left[-\left(0.5k\alpha_{x\ell_2}^{*}s_{2x}^2-D_{x\ell_2}^{*}s_{2x}\right)\right]\exp\left[-\left(0.5k\alpha_{y\ell_2}^{*}s_{2y}^2-D_{y\ell_2}^{*}s_{2y}\right)\right]\qquad\text{in Cartesian coordinates}\qquad(5.9)$$

First tackle the Cartesian coordinate expression, for this, use 3.323.2 of of Gradshteyn and Ryzik 2007, pp. 337 (also given in 3.7)

$$\int_{-\infty}^{\infty}dx\exp\left(-p^2x^2\pm qx\right)=\frac{\pi^{0.5}}{p}\exp\left(\frac{q^2}{4p^2}\right)\qquad(5.10)$$

After inserting (PC9) into (PC4) and arranging the integration over s_{1x} this way collecting the p^2 and q terms so that the appearance is like in (5.10), we get (only relevant terms are shown)

$$I_{s_{1x}}=\int_{-\infty}^{\infty}ds_{1x}\exp\left[-\overbrace{\left(0.5k\alpha_{x\ell_1}+\frac{1}{2\sigma_s^2}-\frac{jk}{2L}\right)}^{p^2}s_{1x}^2+\overbrace{\left(\frac{s_{2x}}{\sigma_s^2}-\frac{jk}{L}r_{1x}+D_{x\ell_1}\right)}^{q}s_{1x}\right]=\left(\frac{2\pi\sigma_s^2L}{k\alpha_{x\ell_1}\sigma_s^2L+L-jk\sigma_s^2}\right)^{0.5}\exp\left[\frac{\left(s_{2x}L-jkr_{1x}\sigma_s^2+D_{x\ell_1}\sigma_s^2L\right)^2}{2\sigma_s^2L\left(k\alpha_{x\ell_1}\sigma_s^2L+L-jk\sigma_s^2\right)}\right]\qquad(5.11)$$

Comparing (5.10) and (5.11), we have identified in (5.11), $p^2 = \dfrac{0.5\left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)}{\sigma_s^2 L}$ and $q = \dfrac{s_{2x}L - jkr_{1x}\sigma_s^2 + D_{x\ell_1}\sigma_s^2 L}{\sigma_s^2 L}$

In (5.11), the first term on the RHS is independent of other integration variables and hence can be taken outside the quadruple integral. The second term, i.e. the exp term on the RHS of (5.11) creates terms for integration over s_{2x}, together with the other terms of the Huygens Fresnel integral, the integration over s_{2x} will appear as

$$I_{s_{2x}} = \int_{-\infty}^{\infty} ds_{2x} \exp\left\{-\left[0.5k\alpha_{x\ell_2}^* + \frac{1}{2\sigma_s^2} + \frac{jk}{2L} - \frac{L}{2\sigma_s^2\left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)}\right]s_{2x}^2\right\} \exp\left[\left(\frac{jkr_{2x}}{L} - \frac{jkr_{1x} - D_{x\ell_1}L}{k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2} + D_{x\ell_2}^*\right)s_{2x}\right] \qquad (5.12)$$

(5.12) can again be solved using (5.10). In this manner p^2 and q terms will become

$$p^2 = \frac{0.5k\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2 L^2 + \left(\alpha_{x\ell_1} + \alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1} - \alpha_{x\ell_2}^*\right)k\sigma_s^2 L + k\sigma_s^2\right]}{L\left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)}$$

$$q = \frac{-j\left(r_{1x} - r_{2x}\right)kL + j\left(\alpha_{x\ell_1}L - j\right)k^2 r_{2x}\sigma_s^2 + D_{x\ell_1}L^2 + \left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)D_{x\ell_2}^* L}{L\left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)} \qquad (5.13)$$

Then $I_{s_{2x}}$ will be

$$I_{s_{2x}} = \left\{\frac{\pi L\left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)}{0.5k\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2 L^2 + \left(\alpha_{x\ell_1} + \alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1} - \alpha_{x\ell_2}^*\right)k\sigma_s^2 L + k\sigma_s^2\right]}\right\}^{0.5}$$
$$\times \exp\left\{\frac{\left[-j\left(r_{1x} - r_{2x}\right)kL + j\left(\alpha_{x\ell_1}L - j\right)k^2 r_{2x}\sigma_s^2 + D_{x\ell_1}L^2 + \left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)D_{x\ell_2}^* L\right]^2}{2kL\left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2 L^2 + \left(\alpha_{x\ell_1} + \alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1} - \alpha_{x\ell_2}^*\right)k\sigma_s^2 L + k\sigma_s^2\right]}\right\} \qquad (5.14)$$

With this step, the integration is more or less completed, since the y part will be identical. All that is left is the rearrangement of terms. Below we do this

$$\text{AmpFac} = \frac{k\sigma_s^2}{\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2L^2 + \left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2L + k\sigma_s^2\right]^{0.5}\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2L^2 + \left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)L^2 + j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)k\sigma_s^2L + k\sigma_s^2\right]^{0.5}} \quad (5.15)$$

Then the exponential term resulting from s_{1x}, s_{2x} integrations will be (for *x* part only)

$$\text{ExpT} = \exp\left(\frac{0.5}{L\left(k\alpha_{x\ell_1}\sigma_s^2L + L - jk\sigma_s^2\right)}\left\{\left(-jkr_{1x} + D_{x\ell_1}L\right)^2\sigma_s^2 + \frac{\left[-j\left(r_{1x}-r_{2x}\right)kL + j\left(\alpha_{x\ell_1}L - j\right)k^2r_{2x}\sigma_s^2 + D_{x\ell_1}L^2 + \left(k\alpha_{x\ell_1}\sigma_s^2L + L - jk\sigma_s^2\right)D_{x\ell_2}^*L\right]^2}{k\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2L^2 + \left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2L + k\sigma_s^2\right]}\right\}\right) \quad (5.16)$$

With other rearrangements, the whole expression for $\Gamma_r \; r_{1x}, r_{1y}, r_{2x}, r_{2y}, L$ of sinusoidal hyperbolic Gaussian beam will be

$$\begin{aligned}
&\Gamma_r\left(r_{1x}, r_{1y}, r_{2x}, r_{2y}, L\right) = k\sigma_s^2 \exp\left[j\frac{k}{2L}\left(r_{1x}^2 - r_{2x}^2 + r_{1y}^2 - r_{2y}^2\right)\right]\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N} \\
&\frac{A_{\ell_1}A_{\ell_2}^*}{\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2L^2 + \left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2L + k\sigma_s^2\right]^{0.5}\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2L^2 + \left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)L^2 + j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)k\sigma_s^2L + k\sigma_s^2\right]^{0.5}} \\
&\exp\left(\frac{0.5}{L\left(k\alpha_{x\ell_1}\sigma_s^2L + L - jk\sigma_s^2\right)}\left\{\left(-jkr_{1x} + D_{x\ell_1}L\right)^2\sigma_s^2 + \frac{\left[-j\left(r_{1x}-r_{2x}\right)kL + j\left(\alpha_{x\ell_1}L - j\right)k^2r_{2x}\sigma_s^2 + D_{x\ell_1}L^2 + \left(k\alpha_{x\ell_1}\sigma_s^2L + L - jk\sigma_s^2\right)D_{x\ell_2}^*L\right]^2}{k\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2L^2 + \left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2L + k\sigma_s^2\right]}\right\}\right) \\
&\exp\left(\frac{0.5}{L\left(k\alpha_{y\ell_1}\sigma_s^2L + L - jk\sigma_s^2\right)}\left\{\left(-jkr_{1y} + D_{y\ell_1}L\right)^2\sigma_s^2 + \frac{\left[-j\left(r_{1y}-r_{2y}\right)kL + j\left(\alpha_{y\ell_1}L - j\right)k^2r_{2y}\sigma_s^2 + D_{y\ell_1}L^2 + \left(k\alpha_{y\ell_1}\sigma_s^2L + L - jk\sigma_s^2\right)D_{y\ell_2}^*L\right]^2}{k\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2L^2 + \left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)L^2 + j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)k\sigma_s^2L + k\sigma_s^2\right]}\right\}\right) \quad (5.17)
\end{aligned}$$

Exercise 5.2 : 1) By setting $r_{1x} = r_{2x} = r_x, r_{1y} = r_{2y} = r_y$ show that at the limit of $N \rightarrow 1$ (5.17) reduces to the Gaussian beam and at $\sigma_s \rightarrow \infty$ to the sinusoidal hyperbolic Gaussian beam given in Q1 of ECE 635 MT dated 22.11.2011

2) Using ParCoh_SinoHypR.m file, test the formulation given in (5.17) and plot receiver intensity graphs and see if they agree with those given by Transmittance635 and Genbeam_receiver_Lagu4 for the same source and propagation parameter set.

For the fully coherent sinusoidal hyperbolic Gaussian given in (5.7), the Cartesian coordinate form of receiver field is found to be

$$\begin{aligned} u_r\left(r_x, r_y, L\right) = \exp\left(jkL\right) \sum_{\ell=1}^{N} \frac{A_\ell}{\left(1+j\alpha_{x\ell}L\right)^{0.5}\left(1+j\alpha_{y\ell}L\right)^{0.5}} \exp\left(-\frac{0.5k\alpha_{x\ell}r_x^2}{1+j\alpha_{x\ell}L}\right) \exp\left(\frac{D_{x\ell}r_x}{1+j\alpha_{x\ell}L}\right) \exp\left[\frac{0.5\,jD_{x\ell}^2L}{k\left(1+j\alpha_{x\ell}L\right)}\right] \\ \times \exp\left(-\frac{0.5k\alpha_{y\ell}r_y^2}{1+j\alpha_{y\ell}L}\right) \exp\left(\frac{D_{y\ell}r_y}{1+j\alpha_{y\ell}L}\right) \exp\left[\frac{0.5\,jD_{y\ell}^2L}{k\left(1+j\alpha_{y\ell}L\right)}\right] \end{aligned} \tag{5.18}$$

Explanation on parameter arrangements in ParCoh_SinoHypR.m and Transmittance635 and Genbeam_receiver_Lagu4

In these files, numeric values of source and propagation parameters are written in the form of arrays or matrices. For instance if Dmat = [0 25 50 200 250] then the writing of Dxarr = [Dmat(2) –Dmat(2)] means that we have set the displacement parameter values for this specific beam as $D_{x1} = 25 \text{ m}^{-1}$, $D_{x2} = -25 \text{ m}^{-1}$. The other arrangement in these files is that in matrices, each line represents a different beam. Bearing in mind that in (5.6) and (5.7) in the definition of Sinusoidal Hyperbolic Gaussian beams, only two summation terns are involved, i.e. the upper limit of the summation, $N = 2$, then the matrices are usually in the form of 6 (covering Gaussian, cosh Gaussian, cos Gaussian, sinh Gaussian, sin Gaussian and annular Gaussian beams) rows and 2 (for the two terms of the summation) columns.

Representation in Matlab

$$\text{Dxarr} = \begin{array}{c} \begin{array}{cc} D_{x1} & \qquad D_{x2} \end{array} \\ \left[\begin{array}{cc} \text{j*Dmat}(1) & -\text{j*Dmat}(1) \\ \text{j*Dmat}(4) & -\text{j*Dmat}(4) \\ \text{1*Dmat}(4) & -\text{1*Dmat}(4) \\ \text{1*Dmat}(1) & -\text{1*Dmat}(1) \end{array}\right] \end{array} = \left[\begin{array}{cc} 0 & 0 \\ 200j & -200j \\ 200 & -200 \\ 0 & 0 \end{array}\right] \begin{array}{l} \leftarrow \text{Gaussian or annular Gaussian beam} \\ \leftarrow \text{cos or sine Gaussian beam} \\ \leftarrow \text{cosh or sinh Gaussian beam} \\ \leftarrow \text{Gaussian or annular Gaussian beam} \end{array} \tag{5.19}$$

To further determine the beams in (5.19), we need to look at the amplitude coefficients (ALarr in Matlab) and source sizes (alfasxarr in Matlab). The following table gives a summary

	Beam name					
Parameter	Gaussian	Annular Gaussian	Cosh Gaussian	Sinh Gaussian	Cos Gaussian	Sine Gaussian
A_1	1	1	1	1	1	j
A_2	0	≥ -1	1	-1	1	-j
D_{x1}	0	0	>0 real	>0 real	>0 imaginary	>0 imaginary
D_{x2}	0	0	<0 real	<0 real	<0 imaginary	<0 imaginary
α_{sx1}	α_{sx}	α_{sx1}	α_{sx1}	α_{sx1}	α_{sx1}	α_{sx1}
α_{sx2}		$<\alpha_{sx1}$	α_{sx1}	α_{sx1}	α_{sx1}	α_{sx1}