Solution to Exercise 5.1 in ECE 635 **HTE – 31.12.2013**

$$\Gamma_r\left(r_{1x},r_{1y},r_{2x},r_{2y},L\right)=k\sigma_s^2\exp\left[j\frac{k}{2L}\left(r_{1x}^2-r_{2x}^2+r_{1y}^2-r_{2y}^2\right)\right]\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}$$

$$\frac{A_{\ell_1}A_{\ell_1}^*}{\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2L^2+\left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2L+k\sigma_s^2\right]^{0.5}\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2L^2+\left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)k\sigma_s^2L+k\sigma_s^2\right]^{0.5}}$$

$$\exp\left(\frac{0.5}{L\left(k\alpha_{x\ell_1}\sigma_s^2L+L-jk\sigma_s^2\right)}\left\{\left(-jkr_{1x}+D_{x\ell_1}L\right)^2\sigma_s^2+\frac{\left[-j\left(r_{1x}-r_{2x}\right)kL+j\left(\alpha_{x\ell_1}L-j\right)k^2r_{2x}\sigma_s^2+D_{x\ell_1}L^2+\left(k\alpha_{x\ell_1}\sigma_s^2L+L-jk\sigma_s^2\right)D_{x\ell_2}^*L\right]^2}{k\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2L^2+\left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)k\sigma_s^2L+k\sigma_s^2\right]}\right\}\right)$$

$$\exp\left(\frac{0.5}{L\left(k\alpha_{y\ell_1}\sigma_s^2L+L-jk\sigma_s^2\right)}\left\{\left(-jkr_{1y}+D_{y\ell_1}L\right)^2\sigma_s^2+\frac{\left[-j\left(r_{1y}-r_{2y}\right)kL+j\left(\alpha_{y\ell_1}L-j\right)k^2r_{2y}\sigma_s^2+D_{y\ell_1}L^2+\left(k\alpha_{y\ell_1}\sigma_s^2L+L-jk\sigma_s^2\right)D_{y\ell_2}^*L\right]^2}{k\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2L^2+\left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)k\sigma_s^2L+k\sigma_s^2\right]}\right\}\right) \quad (1.1)$$

Exercise 5.1 : 1) By setting $r_{1x}=r_{2x}=r_x, r_{1y}=r_{2y}=r_y$ show that at the limit of $N\rightarrow 1$ (1.1) reduces to the Gaussian beam and at $\sigma_s\rightarrow\infty$ to the sinusoidal hyperbolic Gaussian beam given in Q1 of ECE 635 MT dated 22.11.2011

2) Using ParCoh_SinoHypR.m file, test the formulation given in (1.1) and plot receiver intensity graphs and see if they agree with those given by Transmittance635 and Genbeam_receiver_Lagu4 for the same source and propagation parameter set.

For the fully coherent sinusoidal hyperbolic Gaussian beam, the Cartesian coordinate form of receiver field is found to be

$$u_r\left(r_x,r_y,L\right)=\exp\left(jkL\right)\sum_{\ell=1}^{N}\frac{A_\ell}{\left(1+j\alpha_{x\ell}L\right)^{0.5}\left(1+j\alpha_{y\ell}L\right)^{0.5}}\exp\left(-\frac{0.5k\alpha_{x\ell}r_x^2}{1+j\alpha_{x\ell}L}\right)\exp\left(\frac{D_{x\ell}r_x}{1+j\alpha_{x\ell}L}\right)\exp\left[\frac{0.5jD_{x\ell}^2L}{k\left(1+j\alpha_{x\ell}L\right)}\right]$$

$$\exp\left(-\frac{0.5k\alpha_{y\ell}r_y^2}{1+j\alpha_{y\ell}L}\right)\exp\left(\frac{D_{y\ell}r_y}{1+j\alpha_{y\ell}L}\right)\exp\left[\frac{0.5jD_{y\ell}^2L}{k\left(1+j\alpha_{y\ell}L\right)}\right] \quad (1.2)$$

Solution : 1) Initially after setting $r_{1x} = r_{2x} = r_x, r_{1y} = r_{2y} = r_y$ in (1.1), we get

$$I_r\left(r_x, r_y\right) = \Gamma_r\left(r_{1x} = r_{2x}, r_{1y} = r_{2y}, L\right) = k\sigma_s^2 \sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}$$

$$\frac{A_{\ell_1} A_{\ell_1}^*}{\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2 L^2 + \left(\alpha_{x\ell_1} + \alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1} - \alpha_{x\ell_2}^*\right)k\sigma_s^2 L + k\sigma_s^2\right]^{0.5}\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2 L^2 + \left(\alpha_{y\ell_1} + \alpha_{y\ell_2}^*\right)L^2 + j\left(\alpha_{y\ell_1} - \alpha_{y\ell_2}^*\right)k\sigma_s^2 L + k\sigma_s^2\right]^{0.5}}$$

$$\exp\left(\frac{0.5}{L\left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)}\left\{\left(-jkr_x + D_{x\ell_1}L\right)^2\sigma_s^2 + \frac{\left[j\left(\alpha_{x\ell_1}L - j\right)k^2 r_x\sigma_s^2 + D_{x\ell_1}L^2 + \left(k\alpha_{x\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)D_{x\ell_2}^* L\right]^2}{k\left[k\alpha_{x\ell_1}\alpha_{x\ell_2}^*\sigma_s^2 L^2 + \left(\alpha_{x\ell_1} + \alpha_{x\ell_2}^*\right)L^2 + j\left(\alpha_{x\ell_1} - \alpha_{x\ell_2}^*\right)k\sigma_s^2 L + k\sigma_s^2\right]}\right\}\right)$$

$$\exp\left(\frac{0.5}{L\left(k\alpha_{y\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)}\left\{\left(-jkr_y + D_{y\ell_1}L\right)^2\sigma_s^2 + \frac{\left[j\left(\alpha_{y\ell_1}L - j\right)k^2 r_y\sigma_s^2 + D_{y\ell_1}L^2 + \left(k\alpha_{y\ell_1}\sigma_s^2 L + L - jk\sigma_s^2\right)D_{y\ell_2}^* L\right]^2}{k\left[k\alpha_{y\ell_1}\alpha_{y\ell_2}^*\sigma_s^2 L^2 + \left(\alpha_{y\ell_1} + \alpha_{y\ell_2}^*\right)L^2 + j\left(\alpha_{y\ell_1} - \alpha_{y\ell_2}^*\right)k\sigma_s^2 L + k\sigma_s^2\right]}\right\}\right) \quad (1.3)$$

This way (1.3) becomes the intensity for partially coherent sinusoidal hyperbolic Gaussian beam. To arrive at the equivalent fully coherent beam expression, we let $\sigma_s \to \infty$, thus (1.3) will turn into

$$I_r\left(r_x, r_y\right) = \Gamma_r\left(r_{1x} = r_{2x} = r_x, r_{1y} = r_{2y} = r_y, L\right) = \sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\frac{A_{\ell_1} A_{\ell_1}^*}{\left[\alpha_{x\ell_1}\alpha_{x\ell_2}^* L^2 + j\left(\alpha_{x\ell_1} - \alpha_{x\ell_2}^*\right)L + 1\right]^{0.5}\left[\alpha_{y\ell_1}\alpha_{y\ell_2}^* L^2 + j\left(\alpha_{y\ell_1} - \alpha_{y\ell_2}^*\right)L + 1\right]^{0.5}}$$

$$\exp\left(\frac{0.5}{L\left(\alpha_{x\ell_1}L - j\right)}\left\{k\left(-jr_x + D_{x\ell_1}L/k\right)^2 + \frac{\left[j\left(\alpha_{x\ell_1}L - j\right)kr_x + \left(\alpha_{x\ell_1}L - j\right)D_{x\ell_2}^* L\right]^2}{k\left[\alpha_{x\ell_1}\alpha_{x\ell_2}^* L^2 + j\left(\alpha_{x\ell_1} - \alpha_{x\ell_2}^*\right)L + 1\right]}\right\}\right)$$

$$\exp\left(\frac{0.5}{L\left(\alpha_{y\ell_1}L - j\right)}\left\{\left(-jkr_y + D_{y\ell_1}L\right)^2\sigma_s^2 + \frac{\left[j\left(\alpha_{y\ell_1}L - j\right)kr_y + \left(\alpha_{y\ell_1}L - j\right)D_{y\ell_2}^* L\right]^2}{k\left[\alpha_{y\ell_1}\alpha_{y\ell_2}^* L^2 + j\left(\alpha_{y\ell_1} - \alpha_{y\ell_2}^*\right)L + 1\right]}\right\}\right) \quad (1.4)$$

After simplification, (1.4) becomes

$$I_r\left(r_x,r_y\right)=\sum_{\ell_1=1}^{N}\sum_{\ell_2=1}^{N}\frac{A_{\ell_1}A_{\ell_1}^*}{\left[\alpha_{x\ell_1}\alpha_{x\ell_2}^*L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)L+1\right]^{0.5}\left[\alpha_{y\ell_1}\alpha_{y\ell_2}^*L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)L+1\right]^{0.5}}$$
$$\times\exp\left[\frac{-0.5k\left(\alpha_{x\ell_1}+\alpha_{x\ell_2}^*\right)r_x^2}{\alpha_{x\ell_1}\alpha_{x\ell_2}^*L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)L+1}\right]\exp\left[\frac{-jr_xD_{x\ell_1}\left(\alpha_{x\ell_2}^*L+j\right)+jr_xD_{x\ell_2}^*\left(\alpha_{x\ell_1}L-j\right)}{\alpha_{x\ell_1}\alpha_{x\ell_2}^*L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)L+1}\right]\exp\left\{0.5L\frac{D_{x\ell_1}^2\left(\alpha_{x\ell_2}^*L+j\right)+\left(D_{x\ell_2}^*\right)\left(\alpha_{x\ell_1}L-j\right)}{k\left[\alpha_{x\ell_1}\alpha_{x\ell_2}^*L^2+j\left(\alpha_{x\ell_1}-\alpha_{x\ell_2}^*\right)L+1\right]}\right\}$$
$$\times\exp\left[\frac{-0.5k\left(\alpha_{y\ell_1}+\alpha_{y\ell_2}^*\right)r_y^2}{\alpha_{y\ell_1}\alpha_{y\ell_2}^*L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)L+1}\right]\exp\left[\frac{-jr_yD_{y\ell_1}\left(\alpha_{y\ell_2}^*L+j\right)+jr_yD_{y\ell_2}^*\left(\alpha_{y\ell_1}L-j\right)}{\alpha_{y\ell_1}\alpha_{y\ell_2}^*L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)L+1}\right]\exp\left\{0.5L\frac{D_{y\ell_1}^2\left(\alpha_{y\ell_2}^*L+j\right)+\left(D_{y\ell_2}^*\right)\left(\alpha_{y\ell_1}L-j\right)}{k\left[\alpha_{y\ell_1}\alpha_{y\ell_2}^*L^2+j\left(\alpha_{y\ell_1}-\alpha_{y\ell_2}^*\right)L+1\right]}\right\} \quad (1.5)$$

At this stage we can establish correspondence between (1.5) and (1.2). From (1.2), we get the intensity of fully coherent sinusoidal hyperbolic Gaussian beam as

$$I_r\left(r_x,r_y\right)=u_r\left(r_x,r_y,L\right)u_r^*\left(r_x,r_y,L\right)= \qquad ? \qquad (1.6)$$